

TRANSLATES OF EXPONENTIAL BOX SPLINES AND THEIR RELATED SPACES

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ABSTRACT. Exponential box splines (*EB-splines*) are multivariate compactly supported functions on a regular mesh which are piecewise in a space $\mathcal{H}$ spanned by exponential polynomials. This space can be defined as the intersection of the kernels of certain partial differential operators with constant coefficients.

The main part of this paper is devoted to algebraic analysis of the space $\mathbf{H}$ of all entire functions spanned by the integer translates of an *EB-spline*. This investigation relies on a detailed description of $\mathcal{H}$ and its discrete analog $\mathcal{S}$. The approach taken here is based on the observation that the structure of $\mathcal{H}$ is relatively simple when $\mathcal{H}$ is spanned by pure exponentials while all other cases can be analyzed with the aid of a suitable limiting process. Also, we find it more efficient to apply directly the relevant differential and difference operators rather than the alternative techniques of Fourier analysis. Thus, while generalizing the known theory of polynomial box splines, the results here offer a simpler approach and a new insight towards this important special case.

We also identify and study in detail several types of singularities which occur only for *complex EB-splines*. The first is when the Fourier transform of the *EB-spline* vanishes at some critical points, the second is when $\mathcal{H}$ cannot be embedded in $\mathcal{S}$ and the third is when $\mathbf{H}$ is a proper subspace of $\mathcal{H}$. We show, among others, that each of these three cases is strictly included in its former and they all can be avoided by a refinement of the mesh.

1. Introduction and the main results. The first part of this paper is primarily concerned with the analysis of an s -dimensional space $\mathcal{H}$ which is intimately related to the so-called "exponential box splines". In general $\mathcal{H}$ is the intersection of the kernels of a certain set of partial differential operators with constant coefficients. In the univariate case this set contains one ordinary differential operator. We prove that $\mathcal{H}$ is always finite dimensional and is spanned by products of polynomials by exponentials so that in this sense the multivariate setting generalizes the univariate case. In addition to the continuous case described above we investigate a sequence space $\mathcal{S}$ which is the discrete analogue of $\mathcal{H}$. This space is defined by the parallel partial difference operators.

In the second part of the paper we give a detailed algebraic analysis of exponential box splines, based on the results on $\mathcal{H}$ and $\mathcal{S}$, focusing on the space of entire functions spanned by the integer translates of an exponential box spline and the pre-image of this space. The analysis is carried out with an intensive aid of the

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differential and difference operators mentioned above and essentially avoids Fourier analysis which is usually applied to similar problems.

In the sequel we first introduce the exponential box splines and review some of their elementary properties. Then the spaces $\mathcal{H}$ and $\mathcal{S}$ are defined and some of the main results are presented. Finally we give the outline of the paper and conclude the introduction with several notations and conventions.

In the following $|\cdot|$, $\langle \cdot \rangle$, $\cdot$, denote cardinality, linear span and scalar product respectively. The exponential function $\exp(\underline{\theta} \cdot \underline{x})$ is abbreviated as $e_{\underline{\theta}}(\underline{x})$.

Let Γ be a finite set consisting of pairs (not necessarily distinct) of the form

$$(1.1) \quad \gamma = (\underline{x}_\gamma, \lambda_\gamma), \quad \underline{x}_\gamma \in \mathbb{Z}^s \setminus \{0\}, \quad \lambda_\gamma \in \mathbb{C}.$$

Let h be a positive scaling parameter. The exponential box spline based on Γ is a distribution which is defined via its Fourier transform as

$$(1.2) \quad \hat{B}_h(\Gamma|\underline{x}) = \prod_{\gamma \in \Gamma} \int_0^h e^{(\lambda_\gamma - i\underline{x}_\gamma \cdot \underline{x})t} dt.$$

It is known, [R₁], that under the assumption

$$(1.3) \quad \langle \{\underline{x}_\gamma\}_{\gamma \in \Gamma} \rangle = \mathbb{R}^s,$$

$B_h(\Gamma|\cdot)$ is a compactly supported function. Its support is included in (and usually equals) the convex polyhedron

$$(1.4) \quad R_h(\Gamma) := \left\{ \sum_{\gamma \in \Gamma} t_\gamma \underline{x}_\gamma \mid 0 \leq t_\gamma \leq h \ \forall \gamma \in \Gamma \right\}.$$

When $\lambda_\gamma = 0$, all γ , $B_h(\Gamma|\cdot)$ is a piecewise polynomial function and is termed “a polynomial box spline” or simply “a box spline”. Polynomial box splines were introduced by de Boor and DeVore, [BD], and investigated by de Boor and Höllig [BH], Dahmen and Micchelli [DM_{1–3}], Jia [J_{1,2}] and others. The general exponential box splines were introduced in [R_{1,2}].

Let us now present the distribution space and the sequence space which are studied herein. Denote for $\underline{x} \in \mathbb{R}^s$ by $D_{\underline{x}}$ the distributional directional derivative in the $\underline{x}$ -direction and by the $E_{\underline{x}}$ the shift operation defined by

$$(1.5) \quad E_{\underline{x}}f \rightarrow f(\cdot - \underline{x}).$$

Every $K \subset \Gamma$ is associated with the following partial differential and difference operators

$$(1.6) \quad D^K := \prod_{\gamma \in K} (D_{\underline{x}_\gamma} - \lambda_\gamma),$$

$$(1.7) \quad \nabla_h^K := \prod_{\gamma \in K} (I - e^{h\lambda_\gamma} E_{h\underline{x}_\gamma}),$$

where I is the identity mapping. Here D^K acts on $\mathcal{D}'(\mathbb{R}^s)$, the space of all complex valued s -dimensional distributions and ∇_h^K acts on both $\mathcal{D}'(\mathbb{R}^s)$ and space of all sequences defined on the lattice

$$(1.8) \quad \mathbb{Z}_h^s := \{\underline{\alpha} \in \mathbb{R}^s \mid h^{-1}\underline{\alpha} \in \mathbb{Z}^s\}.$$

Now the distribution space corresponding to Γ is defined as

$$(1.9) \quad \mathcal{H}(\Gamma) = \{f \in \mathcal{D}'(\mathbb{R}^s) \mid D^K f \equiv 0, \forall K \subset \Gamma, \exists \langle \{\underline{x}_\gamma\}_{\gamma \in \Gamma \setminus K} \rangle \neq \mathbb{R}^s\},$$

while the discrete space is defined analogously as

$$(1.10) \quad \mathcal{S}_h(\Gamma) = \{u: \mathbb{Z}_h^s \mid \nabla_h^K u \equiv 0, \forall K \subset \Gamma, \exists \langle \{\underline{x}_\gamma\}_{\gamma \in \Gamma \setminus K} \rangle \neq \mathbb{R}^s\}.$$

It can be shown, [R₁], that for every $K \subset \Gamma$ and $f \in \mathcal{D}'(\mathbb{R}^s)$

$$(1.11) \quad \nabla_h^K f = B_h(K) * D^K f.$$

The last formula allows us to produce sequences in $\mathcal{S}_h(\Gamma)$ by restricting elements of $\mathcal{H}(\Gamma) \cap C(\mathbb{R}^s)$ to the lattice points. This close relation between the two spaces is used here to bound the dimension of $\mathcal{H}(\Gamma)$ in terms of the dimension of $\mathcal{S}_h(\Gamma)$.

The following sets, independently introduced in [DM₂] and [J₂], are useful for local analysis of $\mathcal{S}_h(\Gamma)$:

$$(1.12) \quad b_h^\Gamma(\underline{x}) = \{\underline{\alpha} \in \mathbb{Z}_h^s \mid \underline{x} - \underline{\alpha} \in R_h(\Gamma)\}.$$

$b_h^\Gamma(\cdot)$ can be viewed as a map from $\mathbb{R}^s$ to subsets of $\mathbb{Z}_h^s$. Every maximal open set where $b_h^\Gamma(\cdot)$ is constant is termed a Γ - h cell. The union of all cells is denoted by $A_h(\Gamma)$. It is easy to see that $A_h(\Gamma)$ is open and dense in $\mathbb{R}^s$. Also, it is known [R₁], that the exponential box spline $B_h(\Gamma)$ and each of its translates by $\underline{\alpha} \in \mathbb{Z}_h^s$ are infinitely differentiable on $A_h(\Gamma)$.

Another set which plays an important role in the investigation of $\mathcal{H}(\Gamma)$ and $\mathcal{S}_h(\Gamma)$ is the set of all “bases” in Γ

$$(1.13) \quad \mathbf{J}(\Gamma) = \{J \subset \Gamma \mid |J| = s, \langle \{\underline{x}_\gamma\}_{\gamma \in J} \rangle = \mathbb{R}^s\}.$$

Given $J \in \mathbf{J}(\Gamma)$ the notation X_J stands for the $s \times s$ matrix whose columns are $\{\underline{x}_\gamma\}_{\gamma \in J}$.

Since spaces of functions on $\mathbb{R}^s$ which are spanned by exponential polynomials frequently occur in this paper, we find it convenient to refer to such spaces as “PE spaces”. Analogously the restriction of a “PE space” to the lattice $\mathbb{Z}_h^s$ is called “a PE sequence space”.

The main results on $\mathcal{H}(\Gamma)$ and $\mathcal{S}_h(\Gamma)$ are summarized in the following theorem.

THEOREM 1.1. Assume $\langle \{\underline{x}_\gamma\}_{\gamma \in \Gamma} \rangle = \mathbb{R}^s$. Then

- (a) $\mathcal{H}(\Gamma)$ and $\mathcal{S}_h(\Gamma)$ are finite dimensional PE spaces.
- (b) $\dim \mathcal{H}(\Gamma) = |\mathbf{J}(\Gamma)|$.
- (c) $\dim \mathcal{S}_h(\Gamma) = \sum_{J \in \mathbf{J}(\Gamma)} |\det X_J|$.
- (d) For any $\underline{x} \in A_h(\Gamma)$ the set $b_h^\Gamma(\underline{x})$ is unisolvent for interpolation by elements of $\mathcal{S}_h(\Gamma)$ namely any data on $b_h^\Gamma(\underline{x})$ is uniquely interpolated by $\mathcal{S}_h(\Gamma)$.

We also derive a necessary and sufficient condition for the existence of a unique interpolant $f \in \mathcal{H}(\Gamma)$ to every $u(\cdot) \in \mathcal{S}_h(\Gamma)$.

In the case $\lambda_\gamma = 0$ all γ (= the polynomial case), the results stated in Theorem 1.1 are due to Dahmen and Micchelli [DM₁₋₃]. In particular, the polynomial case of Theorem 1.1(b) was proved in [DM₂] by simultaneous induction on $|\Gamma|$ and s . The method taken here is essentially different: we establish first Theorem 1.1 for the “splitting” case (termed *simple*) in which $\mathcal{H}(\Gamma)$ is spanned by pure exponentials. The passage to the general case is executed by a unified limiting process primarily based on the following theorem.

THEOREM 1.2. *Let L be a positive number and n a positive integer. Denote by $E(n, L)$ the set of all s -dimensional functions of the form $\sum_{j=1}^n c_j e_{\theta^j}(\underline{x})$, where $c_1, \dots, c_n$ are complex numbers and the exponents $\theta^1, \dots, \theta^n$ satisfy the following condition: $\max_{1 \leq r \leq s, 1 \leq j \leq n} |\theta_r^j| < L$. Let $\{H_j\}_{j=1}^\infty \subset E(n, L)$ be a sequence of linear spaces. Then there exists a subsequence $\{H_{j_l}\}_{l=1}^\infty$ and a function space H such that*

- (a) $\dim H = \min \dim H_j =: d$.
- (b) Every $f \in H$ is a limit in the $C^\infty(\mathbb{R}^s)$ topology of a sequence $\{f_{j_l} | f_{j_l} \in H_{j_l} \forall l\}$.

The fact that $\mathcal{H}(\Gamma)$ is closed under differentiation (D -invariant) and $\mathcal{S}_h(\Gamma)$ is closed under translates by $\underline{\alpha} \in \mathbb{Z}_h^s$ (shift-invariant) is of fundamental significance here. The following theorem gives a characterization of a *finite dimensional* space which satisfies such invariance properties.

THEOREM 1.3. (a) *Every finite dimensional D -invariant distribution space is a PE space.*

(b) *Every finite dimensional shift-invariant sequence space is a PE sequence space.*

Theorem 1.3(a) can also be stated with D -invariance replaced by translation-invariance. Thus, for the space of *tempered* distributions rather than $\mathcal{D}'(\mathbb{R}^s)$ this result follows directly from the Whitney-Schwartz theorem, [S₂, Theorem VI], which says that every closed translation-invariant space of tempered distributions contains a dense PE subspace. Yet, the analogous results to the Whitney-Schwartz theorem for other spaces are usually not valid. E.g., for $C^\infty(\mathbb{R}^s)$ the claim is true only for $s = 1$ (see [S₁, G]).

In §2 we prove Theorem 1.2 and apply this theorem to show that

(1.14)
$$\dim \mathcal{H}(\Gamma) \geq |\mathbf{J}(\Gamma)|.$$

§3 is devoted to the proof of Theorem 1.3. In §4 it is first shown that $\mathcal{S}_h(\Gamma)$ and $\mathcal{H}(\Gamma)$ are finite dimensional PE spaces. Then their structure is analyzed in detail with special emphasis to the “simple” case. The proof of Theorem 1.1 is completed in §5.

Substituting $f = B_h(\Gamma \backslash K)$ in (1.11) one obtains the identity, [R₁],

(1.15)
$$\nabla_h^K B_h(\Gamma \backslash K) = D^K B_h(\Gamma), \quad \forall K \subset \Gamma.$$

This simple formula is our basic tool in the investigation of exponential box splines made in §6. There we define for $u: \mathbb{Z}_h^s$

(1.16)
$$B_h(\Gamma) *_h u := \sum_{\underline{\alpha} \in \mathbb{Z}_h^s} u(\underline{\alpha}) B_h(\Gamma | \cdot - \underline{\alpha}),$$

and view $B_h(\Gamma) *_h$ as an operation from the space of all sequences defined on $\mathbb{Z}_h^s$. Our aim is to characterize the space $\mathbf{H}_h(\Gamma)$ of all entire functions in the range of $B_h(\Gamma) *_h$. The main relations between $\mathbf{H}_h(\Gamma)$, $\mathcal{H}(\Gamma)$, $\mathcal{S}_h(\Gamma)$ and $B_h(\Gamma) *_h$ are as follows.

THEOREM 1.4.

- (1.17)
 - (a) $\mathbf{H}_h(\Gamma) \subset \mathcal{H}(\Gamma)$.
 - (b) $B_h(\Gamma) *_h$ maps $\mathcal{S}_h(\Gamma)$ onto $\mathbf{H}_h(\Gamma)$.

The analysis of $\mathbf{H}_h(\Gamma)$ is completed in §7 where we establish necessary and sufficient conditions for equality in (1.17) and derive a formula for $\dim \mathbf{H}_h(\Gamma)$. To describe some of the results of that section denote:

$$(1.18) \quad \Theta(\Gamma) = \{\underline{\theta} \in \mathbb{C}^s \mid \exists J \in \mathbf{J}(\Gamma), \exists \underline{x}_\gamma \cdot \underline{\theta} - \lambda_\gamma = 0 \ \forall \gamma \in J\}.$$

Then we have

THEOREM 1.5. *Consider the following three conditions:*

- (a) $B_h(\Gamma - i\underline{\theta}) \neq 0, \ \forall \underline{\theta} \in \Theta(\Gamma).$
- (b) $h(\underline{\theta}^1 - \underline{\theta}^2) \notin 2\pi i\mathbb{Z}^s \setminus \underline{0}, \ \forall \underline{\theta}^1, \underline{\theta}^2 \in \Theta(\Gamma).$
- (c) $\mathbf{H}_h(\Gamma) = \mathcal{H}(\Gamma).$

Then (a) $\Rightarrow$ (b) $\Rightarrow$ (c).

It is also proved that in the *univariate* case conditions (b), (c) of Theorem 1.5 are equivalent. Yet, with the aid of a bivariate counterexample we show that in the multivariate case the implication (c) $\Rightarrow$ (b) is not valid.

In this paper we consistently use standard multivariate notations. Thus for $\underline{\alpha} \in \mathbb{Z}^s$, $\underline{x} \in \mathbb{R}^s$ we use $|\underline{\alpha}| = \alpha_1 + \dots + \alpha_s$, $D^{(\underline{\alpha})} = \partial^{|\underline{\alpha}|} / \partial x_1^{\alpha_1} \dots \partial x_s^{\alpha_s}$ and $\underline{x}^{\underline{\alpha}} = \prod_{j=1}^s x_j^{\alpha_j}$. The space of all s -dimensional polynomials is denoted as usual by π . Also we find it convenient to denote for every $K \subset \Gamma$

$$X_K := \{\underline{x}_\gamma\}_{\gamma \in K}, \quad \underline{\lambda}_K := \{\lambda_\gamma\}_{\gamma \in K}, \quad \langle K \rangle := \langle X_K \rangle.$$

A vector $\underline{x}_\gamma$ (with $\gamma \in \Gamma$) is sometimes treated as the pair $(\underline{x}_\gamma, 0)$. So $D^{X_K} = \prod_{\gamma \in K} D_{\underline{x}_\gamma}$, $B_h(X_K)$ is the (polynomial) box spline based on $(X_K, \underline{0})$ etc. When concerning a set of a single element we usually omit brackets; thus D^γ , $\nabla_h^{\underline{x}_\gamma}$, $\Gamma \setminus \gamma$ are used rather than the rigorous notations $D^{\{\gamma\}}$, $\nabla_h^{\{\underline{x}_\gamma\}}$, $\Gamma \setminus \{\gamma\}$. Whenever it is possible we may identify a function space with its restriction to the lattice points. Finally, to simplify the definition of $\mathcal{H}(\Gamma)$ and $\mathcal{S}_h(\Gamma)$ we denote

$$(1.19) \quad \mathcal{H}(\Gamma) := \{K \subset \Gamma \mid \langle \Gamma \setminus K \rangle \neq \mathbb{R}^s\}.$$

With the aid of this notation we may write

$$\mathcal{H}(\Gamma) = \bigcap_{K \in \mathcal{H}(\Gamma)} \ker D^K, \quad \mathcal{S}_h(\Gamma) = \bigcap_{K \in \mathcal{H}(\Gamma)} \ker \nabla_h^K.$$

2. A lower bound for $\dim \mathcal{H}(\Gamma)$. We make here the first step in the analysis of $\mathcal{H}(\Gamma)$ and $\mathcal{S}_h(\Gamma)$ by showing that

$$(2.1) \quad \dim \mathcal{H}(\Gamma) \geq |\mathbf{J}(\Gamma)|.$$

In case $\underline{\lambda}_\Gamma = \underline{0}$, this result is Theorem 2.2 in [DM₂] which was proved there by induction on $|\Gamma|$ and s . The method used here differs from that of [DM₂]. First we prove (2.1) for “simple” defining sets and then apply Theorem 1.2 to extend the results to general Γ ’s. The proof of Theorem 1.2 is postponed to the end of this section.

Given a defining set Γ and $\underline{\theta} \in \mathbb{C}^s$ denote

$$(2.2) \quad \Gamma_{\underline{\theta}} = \{\gamma \in \Gamma \mid \underline{x}_\gamma \cdot \underline{\theta} - \lambda_\gamma = 0\},$$

and define (compare with (1.18))

$$(2.3) \quad \Theta(\Gamma) = \{\underline{\theta} \in \mathbb{C}^s \mid \langle \Gamma_{\underline{\theta}} \rangle = \mathbb{R}^s\}.$$

The elements of $\Theta(\Gamma)$ will be referred to as “nodes”. Note that for $\gamma \in \Gamma$ and $\underline{\theta} \in \mathbb{C}^s$, $D^\gamma e_{\underline{\theta}}(\cdot) \equiv 0$ if and only if $\gamma \in \Gamma_{\underline{\theta}}$. Now, for $\underline{\theta} \in \Theta(\Gamma)$ and $K \in \mathcal{K}(\Gamma)$ one has by definition $\langle \Gamma \setminus K \rangle \neq \mathbb{R}^s$ and consequently (2.3) implies $K \cap \Gamma_{\underline{\theta}} \neq \emptyset$. Thus

$$D^K e_{\underline{\theta}} = D^{K \setminus \Gamma_{\underline{\theta}}} (D^{K \cap \Gamma_{\underline{\theta}}} e_{\underline{\theta}}) \equiv 0,$$

and we conclude

PROPOSITION 2.1. *For each $\underline{\theta} \in \Theta(\Gamma)$, $e_{\underline{\theta}}(\cdot) \in \mathcal{K}(\Gamma)$. In particular $\dim \mathcal{K}(\Gamma) \geq |\Theta(\Gamma)|$.*

Let $J \in \mathbf{J}(\Gamma)$; since J is a “basis” it determines a *unique* $\underline{\theta} \in \Theta(\Gamma)$ for which $\underline{\theta} \cdot \underline{x}_\gamma - \lambda_\gamma = 0$, $\forall \gamma \in J$. Moreover, every $\underline{\theta} \in \Theta(\Gamma)$ can be obtained in this way because $\langle \Gamma_{\underline{\theta}} \rangle = \mathbb{R}^s$ and thus contains some $J \in \mathbf{J}(\Gamma)$. Hence we conclude

$$(2.4) \quad |\mathbf{J}(\Gamma)| \geq |\Theta(\Gamma)|.$$

Equality in (2.4) does not hold in general. One may have several J ’s which correspond to the same $\underline{\theta} \in \Theta(\Gamma)$. E.g., in case $\underline{\lambda}_\Gamma = \underline{0}$ the only node in $\Theta(\Gamma)$ is $\underline{0}$. In order that equality holds in (2.4), it is necessary and sufficient that for each $\underline{\theta} \in \Theta(\Gamma)$ the set $\Gamma_{\underline{\theta}}$ will contain exactly s elements (namely $\Gamma_{\underline{\theta}}$ will be a “basis” from $\mathbf{J}(\Gamma)$).

DEFINITION 2.1. We say that Γ is a *simple defining set* if $|\mathbf{J}(\Gamma)| = |\Theta(\Gamma)|$.

In a simple case Proposition 2.1 can be restated as follows.

PROPOSITION 2.2. *Assume Γ is simple. Then $\dim \mathcal{K}(\Gamma) \geq |\mathbf{J}(\Gamma)|$.*

In order to extend this last result to general defining sets, we need to know how often one encounters simple defining sets. This question was answered in $[\mathbf{R}_1]$ in the following way.

PROPOSITION 2.3. *Let X be a set of n nontrivial vectors in $\mathbb{R}^s$. Assume $\langle X \rangle = \mathbb{R}^s$. For each $\underline{\lambda} \in \mathbb{C}^n$, denote by $\Gamma_{\underline{\lambda}}$ the defining set composed of $(X, \underline{\lambda})$. Then the set $\{\underline{\lambda} \in \mathbb{C}^n \mid \Gamma_{\underline{\lambda}} \text{ is simple}\}$ is open and dense in $\mathbb{C}^n$. Its complement is the union of finitely many hyperplanes of $\mathbb{C}^n$.*

Now, we combine Theorem 1.2 together with Propositions 2.2 and 2.3 to derive

THEOREM 2.1. *For every defining set Γ $\dim \mathcal{K}(\Gamma) \geq |\mathbf{J}(\Gamma)|$.*

PROOF. Let Γ be fixed. Proposition 2.3 ensures the existence of *simple* defining sets $\{\Gamma_k\}_{k=1}^\infty$ such that $X_{\Gamma_k} = X_\Gamma$ for all k and $\underline{\lambda}_{\Gamma_k} \xrightarrow[k \rightarrow \infty]{} \underline{\lambda}_\Gamma$. By Proposition 2.2

$$\dim \mathcal{K}(\Gamma_k) \geq |\mathbf{J}(\Gamma_k)| = |\mathbf{J}(\Gamma)|,$$

for all k . Also since $\{X_{\Gamma_k}\}_{k=1}^\infty$ are all equal and $\{\underline{\lambda}_{\Gamma_k}\}_{k=1}^\infty$ is convergent, it follows that $\bigcup_{k=1}^\infty \Theta(\Gamma_k)$ is bounded, say by $L > 0$. Now if $f_k \in \mathcal{K}(\Gamma_k)$ and $f_k \xrightarrow[k \rightarrow \infty]{} f$ in $C^\infty(\mathbb{R}^s)$ topology then for every $K \in \mathcal{K}(\Gamma)$

$$D^K f = \lim_{k \rightarrow \infty} D^{K_k} f_k = 0$$

where $\{K_k\}_{k=1}^\infty$ are the corresponding sets in $\{\mathcal{K}(\Gamma_k)\}_{k=1}^\infty$, and thus $f \in \mathcal{K}(\Gamma)$. Application of Theorem 1.2 with n replaced by $|\mathbf{J}(\Gamma)|$ completes the proof. $\square$

We now turn to the proof of Theorem 1.2. This proof is primarily based on the following lemma.

LEMMA 2.1. For each $k \in \mathbb{N}$, $L > 0$ and compact $M \subset \mathbb{R}^s$ there exists a constant $c = c(k, L, M)$ such that if $f \in E(k, L)$ and

$$(2.5) \quad |D^{(\underline{\alpha})} f(\underline{0})| \leq \varepsilon, \quad 0 \leq |\underline{\alpha}| < k,$$

then

$$(2.6) \quad |f(\underline{x})| \leq \varepsilon c, \quad \forall \underline{x} \in M.$$

PROOF. Clearly we can and shall assume $\varepsilon = 1$. Define $R = R(M) = \sup_{\underline{x} \in M} \|\underline{x}\|_\infty$. Fix $\underline{x}^0 \in M$ and choose $\underline{y} \in \mathbb{R}^s$ and $0 \leq t_0 < R$ such that $\|\underline{y}\|_\infty = 1$ and $\underline{x}^0 = \underline{y}t_0$. Define

$$\varphi(t) = f(t\underline{y}).$$

It follows from (2.5) that

$$(2.7) \quad |\varphi^{(j)}(\underline{0})| = |(D\underline{y})^j f(\underline{0})| \leq s^j \leq s^k, \quad j = 0, \dots, k-1.$$

Also, since $f \in E(k, L)$ we have that $\varphi(t) = \sum_{j=1}^k c_j e_{\lambda_j}(t)$ where

$$(2.8) \quad |\lambda_j| = |\underline{\theta}^j \cdot \underline{y}| \leq sL.$$

Therefore, $\varphi(t)$ lies in the kernel of

$$\prod_{j=1}^k (D - \lambda_j) = D^k + a_{k-1}D^{k-1} + \dots + a_0,$$

where $D = d/dt$ and the coefficients $a_0, \dots, a_{k-1}$ satisfy

$$(2.9) \quad |a_j| < L_1 := 2^k(1 + sL)^k.$$

Writing this differential operator in the usual matrix form we have

$$(2.10) \quad \begin{pmatrix} \varphi' \\ \varphi^{(2)} \\ \vdots \\ \varphi^{(k)} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & & \vdots \\ 0 & \cdot & \cdot & \dots & 1 \\ -a_0 & \cdot & \cdot & \dots & -a_{k-1} \end{pmatrix} \begin{pmatrix} \varphi \\ \varphi' \\ \vdots \\ \varphi^{(k-1)} \end{pmatrix}$$

Thus denoting the matrix in (2.10) by A we find that

$$(2.11) \quad \varphi(t_0) = (1, 0, \dots, 0)e^{At_0} \begin{pmatrix} \varphi(0) \\ \varphi'(0) \\ \vdots \\ \varphi^{(k-1)}(0) \end{pmatrix}.$$

From (2.9) we infer that $\|e^{At_0}\|_\infty \leq e^{kL_1 t_0}$, so (2.7) together with (2.11) yields

$$|f(\underline{x}^0)| = |\varphi(t_0)| \leq ks^k e^{kL_1 t_0} \leq ks^k e^{kL_1 R}.$$

Since $\underline{x}^0 \in M$ was arbitrary the proof is completed by setting

$$c = ks^k e^{[k2^k(1+sL)^k]R(M)}. \quad \square$$

The following result easily derived from Lemma 2.1 will be the crux in the proof of Theorem 1.2.

COROLLARY 2.1. Let $\{f_j(\underline{x})\}_{j=1}^\infty \subset E(n, L)$. Assume

$$(2.12) \quad D^{(\underline{\alpha})} f_j(\underline{0}) \xrightarrow{j \rightarrow \infty} a_{\underline{\alpha}} \in \mathbb{C}, \quad 0 \leq |\underline{\alpha}| < 2n.$$

Then $\{f_j\}_{j=1}^\infty$ is convergent in the $C^\infty(\mathbb{R}^s)$ topology.

PROOF. First we identify $\mathbb{C}^s$ with $\mathbb{R}^{2s}$ in the usual way

$$(z_1, \dots, z_s) \sim (x_1 + ix_{s+1}, \dots, x_s + ix_{2s}).$$

Every $f \in E(n, L)$ extends to an entire function which we denote by $\tilde{f}$. Then

$$(2.13) \quad \frac{\partial \tilde{f}}{\partial z_j} = -i \frac{\partial \tilde{f}}{\partial x_{j+s}} = \frac{\partial \tilde{f}}{\partial x_j} = \frac{\partial f}{\partial x_j}, \quad j = 1, \dots, s.$$

By (2.12) we know that $\{D^{(\underline{\alpha})} f_j(\underline{0})\}_{j=1}^\infty$ is a Cauchy sequence for every $0 \leq |\underline{\alpha}| < 2n$. From (2.13) we conclude that $\{D^{(\underline{\alpha})} \tilde{f}(\underline{0})\}$, $0 \leq |\underline{\alpha}| < 2n$, is also a Cauchy sequence (namely when the derivatives may be taken with respect to *all* variables $(x_1, \dots, x_{2s})$). Thus by application of Lemma 2.1 to $\tilde{f}_j - \tilde{f}_m$ (with s replaced by $2s$ and $k = 2n$), we find that for every compact $M \subset \mathbb{C}^s$ and $\varepsilon > 0$ there exists j_0 such that for every $j, m > j_0$ and $\underline{x} \in M$

$$|\tilde{f}_j(\underline{x}) - \tilde{f}_m(\underline{x})| < \varepsilon.$$

Hence $\{\tilde{f}_j\}_{j=1}^\infty$ is a Cauchy sequence in $H(\mathbb{C}^s)$ and therefore $\{f_j\}_{j=1}^\infty$ is a Cauchy sequence in $C^\infty(\mathbb{R}^s)$. Now the corollary follows from the completeness of $C^\infty(\mathbb{R}^s)$. $\square$

PROOF OF THEOREM 1.2 Let $m = \binom{2n+s-1}{s}$ be the number of all $\underline{\alpha} \in \mathbb{Z}_+^s$ with $|\underline{\alpha}| < 2n$. Let χ be a map from $E(n, L)$ to $\mathbb{C}^m$ defined as follows:

$$\chi(f) = \{D^{(\underline{\alpha})} f(\underline{0})\}_{0 \leq |\underline{\alpha}| < 2n}.$$

From Lemma 2.1 we conclude that χ is 1-1 on each H_j ; hence $\chi(H_j)$ is a subspace of $\mathbb{C}^m$ of dimension $\geq d$. So we can choose for each j a set $\{f_{j,1}, \dots, f_{j,d}\} \subset H_j$ such that $\{\chi(f_{j,k})\}_{k=1}^d$ is an orthonormal set in $\mathbb{C}^m$. Since the unit sphere of $\mathbb{C}^m$ is compact we can find a subsequence $\{j_r\}_{r=1}^\infty$ such that $\{\chi(f_{j_r,k})\}_{r=1}^\infty$ is convergent for $k = 1, \dots, d$. By Corollary 2.1 there exist $f_1, \dots, f_d$ in $C^\infty(\mathbb{R}^s)$ such that

$$f_{j_r,k} \xrightarrow{r \rightarrow \infty} f_k, \quad k = 1, \dots, d,$$

in $C^\infty(\mathbb{R}^s)$. Denote $H = \langle f_1, \dots, f_d \rangle$. Let us show that $\dim H = d$ by proving that $f_1, \dots, f_d$ are linearly independent. Assume $\underline{a} \in \mathbb{C}^d$, $\sum_{k=1}^d a_k f_k \equiv 0$ on $\mathbb{R}^s$. Then

$$\lim_{r \rightarrow \infty} \sum_{k=1}^d a_k f_{j_r,k} = 0$$

in $C^\infty(\mathbb{R}^s)$ topology. Since χ is continuous on $C^\infty(\mathbb{R}^s)$ we conclude that

$$\lim_{r \rightarrow \infty} \sum_{k=1}^d a_k \chi(f_{j_r,k}) = 0$$

in $\mathbb{C}^m$. But for each r $\{\chi(f_{j_r,k})\}_{k=1}^d$ is orthonormal and therefore

$$\sum_{k=1}^d |a_k|^2 = \left\| \sum_{k=1}^d a_k \chi(f_{j_r,k}) \right\|_2 \xrightarrow{r \rightarrow \infty} 0.$$

Consequently $\underline{a} = \underline{0}$ and the proof of Theorem 1.2 is completed. $\square$

3. Proof of Theorem 1.3. Before proving Theorem 1.3 we state and prove an apparently weaker result.

LEMMA 3.1. (a) Every finite dimensional D -invariant subspace H of a PE space is also a PE space.

(b) Every finite dimensional shift-invariant subspace S of a PE sequence space is also a PE space.

PROOF. (a) Let $f \in H$. Since f belongs to a PE space it has the form

$$(3.1) \quad f(\underline{x}) = \sum_{j=1}^n p_j(\underline{x}) e_{\underline{\theta}^j}(\underline{x}), \quad p_j \in \pi, \quad \underline{\theta}^j \in \mathbb{C}^s, \quad j = 1, \dots, n.$$

Without loss of generality we can assume that $\{\underline{\theta}^j\}_{j=1}^n$ are pairwise distinct. Thus, for $j = 2, \dots, n$ there exist $\underline{x}^j \in \mathbb{R}^s$ and $\lambda_j \in \mathbb{C}$ such that

$$\underline{x}^j \cdot \underline{\theta}^1 - \lambda_j \neq 0, \quad \underline{x}^j \cdot \underline{\theta}^j - \lambda_j = 0.$$

Denote

$$D^j = D_{\underline{x}^j} - \lambda_j;$$

then for every $p \in \pi$

$$D^j[e_{\underline{\theta}^1}(\cdot)p(\cdot)] = e_{\underline{\theta}^1}(\cdot)q(\cdot), \quad \deg q(\cdot) = \deg p(\cdot),$$

but

$$D^j[e_{\underline{\theta}^j}(\cdot)p(\cdot)] = e_{\underline{\theta}^j}(\cdot)[D_{\underline{x}^j}p(\cdot)].$$

Defining $L_1 = \prod_{j=2}^n (D^j)^{\deg p_j + 1}$, we see that $L_1 f(\cdot) = e_{\underline{\theta}^1}(\cdot)q(\cdot)$ where $\deg q(\cdot) = \deg p_1(\cdot)$. Since H is D -invariant, we know that $L_1 f \in H$. Denote by H_1 the subspace of H consisting of elements of the form $e_{\underline{\theta}^1}(\cdot)p(\cdot)$, $p \in \pi$. Since H_1 is of finite dimension, the preceding arguments show that L_1 is an injective map of H_1 onto itself. Thus, since $q(\cdot)e_{\underline{\theta}^1}(\cdot) \in H_1$ and $L_1(e_{\underline{\theta}^1}(\cdot)p_1(\cdot)) = e_{\underline{\theta}^1}(\cdot)q(\cdot)$ we must have $e_{\underline{\theta}^1}(\cdot)p_1(\cdot) \in H_1 \subset H$. This argument can be applied to any index other than 1; consequently $\{e_{\underline{\theta}^j}(\cdot)p_j(\cdot)\}_{j=1}^n \subset H$ and (a) is established.

The proof of (b) is analogous: if $u \in S$, then since S is included in a PE space, we have

$$(3.2) \quad u(\underline{\alpha}) = \sum_{j=1}^n e_{\underline{\theta}^j}(\underline{\alpha}) p_j(\underline{\alpha}), \quad p_j \in \pi, \quad \underline{\theta}^j \in \mathbb{C}^s, \quad j = 1, \dots, n.$$

Since we take $\underline{\alpha} \in \mathbb{Z}^s$ we can assume $\underline{\theta}^j - \underline{\theta}^k \notin 2\pi i \mathbb{Z}^s$ for $j \neq k$. Given $2 \leq j \leq n$ we can therefore find $\underline{\alpha}^j \in \mathbb{Z}^s$ and $\lambda_j \in \mathbb{C}$ such that

$$e^{\underline{\alpha}^j \cdot \underline{\theta}^1 - \lambda_j} \neq 1, \quad e^{\underline{\alpha}^j \cdot \underline{\theta}^j - \lambda_j} = 1,$$

and denote

$$\Delta_1 = \prod_{j=2}^n (E_{\underline{\alpha}^j} - e^{\lambda_j})^{\deg p_j + 1}.$$

The rest of the proof is identical to that of part (a) with Δ_1 replacing L_1 . $\square$

Next, we reduce the first part of Theorem 1.3 from $\mathcal{D}'(\mathbb{R}^s)$ to $C^\infty(\mathbb{R}^s)$:

THEOREM 3.2. *Let H be a subspace of $\mathcal{D}'(\mathbb{R}^s)$ which is closed under convolution with compactly supported C^∞ functions. Assume $H \cap C^\infty(\mathbb{R}^s)$ is finite dimensional. Then $H \subset C^\infty(\mathbb{R}^s)$.*

PROOF. Denote $N = \dim(H \cap C^\infty(\mathbb{R}^s))$. Let $f_1, \dots, f_{N+1} \in H$. Let $\{\varphi_h\}_{h>0}$ be an infinitely differentiable approximate identity of compact support (see [Ru, p. 157]). Then $\{f_j * \varphi_h\}_{j=1}^{N+1} \subset H \cap C^\infty(\mathbb{R}^s)$, so there exist $\underline{c}_h := \{c_{h,j}\}_{j=1}^{N+1}$ such that

$$\left(\sum_{j=1}^{N+1} c_{h,j} f_j \right) * \varphi_h = \sum_{j=1}^{N+1} c_{h,j} (f_j * \varphi_h) \equiv 0.$$

We can assume $\sum_{j=1}^{N+1} c_{h,j}^2 = 1$ for all h and the compactness of the unit ball in $\mathbb{C}^{N+1}$ implies the existence of a sequence $h_k \xrightarrow{k \rightarrow \infty} 0$ such that $\underline{c}_{h_k} \xrightarrow{k \rightarrow \infty} \underline{c}^\circ \neq 0$.

Therefore

$$\sum_{j=1}^{N+1} c_j^\circ f_j = \lim_{k \rightarrow \infty} \left(\sum_{j=1}^{N+1} c_{h_k,j} f_j \right) * \varphi_{h_k} = 0.$$

Consequently, $\dim H \leq N$ which implies $H = H \cap C^\infty(\mathbb{R}^s)$. $\square$

We now turn to the proof of Theorem 1.3(a). Denote the space under consideration by H . Let $\{\underline{e}^j\}_{j=1}^s$ be the standard basis of $\mathbb{R}^s$. We know that $D_{\underline{e}^j}$ is a linear transformation which maps H into itself. Denote by $D_{\underline{e}^j}|_H$ the restriction of $D_{\underline{e}^j}$ to H and by $(\lambda_{j,1}, \dots, \lambda_{j,n})$ the spectrum of $D_{\underline{e}^j}|_H$ counting (algebraic) multiplicities. Then by the Cayley-Hamilton theorem

$$D^j := \prod_{k=1}^n (D_{\underline{e}^j}|_H - \lambda_{j,k}) = 0, \quad H \subset \bigcap_{j=1}^s \ker D^j =: H_1.$$

Clearly, $H_1 \cap C^\infty(\mathbb{R}^s)$ is a finite dimensional PE space. In fact a basis to $H_1 \cap C^\infty(\mathbb{R}^s)$ is given in terms of the functions $e_{\underline{\lambda}}(\underline{x}) \underline{x}^{\underline{\nu}}$, where for every $1 \leq j \leq s$, λ_j lies in the spectrum of $D_{\underline{e}^j}|_H$ and $\nu_j - 1$ does not exceed the multiplicity of λ_j .

But H_1 is also closed under convolution with test functions. Indeed for $f \in H_1$ and $g \in \mathcal{D}(\mathbb{R}^s)$ one has

$$D^j(f * g) = (D^j f) * g = 0, \quad j = 1, \dots, s.$$

Therefore, application of Lemma 3.2 yields that H_1 is a PE space and hence by Lemma 3.1 so is H . This completes the proof of Theorem 1.3(a).

To prove (b), note that the same arguments used in the proof of part (a) show the existence of $\underline{\lambda}^1, \dots, \underline{\lambda}^s \in \mathbb{C}^n$ such that

$$S \subset \bigcap_{j=1}^s \left(\ker \prod_{k=1}^n (E_{\underline{e}^j} - \lambda_k^j) \right) =: S_1.$$

Since S_1 is clearly a PE sequence space, Lemma 3.1(b) yields the desired result. $\square$

In the analysis of $\mathcal{H}(\Gamma)$ and $\mathcal{S}_h(\Gamma)$ we frequently investigate the restriction of a PE space to the lattice points. When doing so a certain precaution is needed: in case $\underline{\theta}^1 - \underline{\theta}^2 \in 2\pi i \mathbb{Z}^s$ one has

$$e_{\underline{\theta}^1}(\cdot)|_{\mathbb{Z}^s} \equiv e_{\underline{\theta}^2}(\cdot)|_{\mathbb{Z}^s},$$

and therefore a function cannot be identified with its restriction. The following proposition shows that besides this case restriction to $\mathbb{Z}^s$ is “safe”:

PROPOSITION 3.1. Let $\{\underline{\theta}^j\}_{j=1}^n \subset \mathbb{C}^s$, $\underline{\theta}^j - \underline{\theta}^k \notin 2\pi i\mathbb{Z}^s$ for $j \neq k$. Let

$$f_j(\cdot) = e_{\underline{\theta}^j}(\cdot) p_j(\cdot) \quad p_j \in \pi, \quad j = 1, \dots, n.$$

Then $\{f_j(\cdot)\}_{j=1}^n$ are linearly independent on $\mathbb{Z}^s$.

PROOF. Assume $\sum_{j=1}^n f_j(\underline{\alpha}) = 0$, $\forall \underline{\alpha} \in \mathbb{Z}^s$. Since $\underline{\theta}^k - \underline{\theta}^j \notin 2\pi i\mathbb{Z}^s$ for $k \neq j$, one can construct as in the proof of Lemma 3.1(b) a difference operator Δ_1 such that

$$\Delta_1 f_j(\cdot) \equiv 0, \quad j = 2, \dots, n, \quad \Delta_1 f_1(\cdot) = e_{\underline{\theta}^1}(\cdot) q(\cdot),$$

where $\deg q = \deg p_1$. Since $\sum_{j=1}^n f_j$ vanishes on $\mathbb{Z}^s$ we conclude that $q(\cdot)$ vanishes on $\mathbb{Z}^s$. Thus $q(\cdot) \equiv 0$ and hence $p_1(\cdot) = 0$. Application of induction to n completes the proof. $\square$

Note that every $e_{\underline{\theta}^j}(\cdot)|_{\mathbb{Z}^s}$ can be viewed as a complex homomorphism of $\mathbb{Z}^s$. Thus the case $p_j = \text{const}_j$, $j = 1, \dots, n$, could also be obtained by Artin's theorem about the linear independence of homomorphisms (see, e.g., [L, p. 209]).

Finally, we mention the following fact which is easily verified by the methods used in Lemma 3.1.

COROLLARY 3.1. Let F be a D -invariant subspace of a PE distribution space or a shift invariant subspace of a PE sequence space. Then F contains an exponential.

4. The structure of $\mathcal{H}(\Gamma)$ and $\mathcal{S}_h(\Gamma)$. We first show that $\mathcal{S}_h(\Gamma)$ and $\mathcal{H}(\Gamma)$ are of finite dimension, which allows us to deduce from Theorem 1.3 that they are PE spaces. This fact greatly simplifies the analysis of their structure.

Fix $h > 0$ and $J \in \mathbf{J}(\Gamma)$ and denote

$$(4.1) \quad G_h^J = \{X_J \underline{\alpha} \mid \underline{\alpha} \in \mathbb{Z}_h^s\}.$$

Since $X_J \subset \mathbb{Z}^s$ we know that G_h^J is a subgroup of $\mathbb{Z}_h^s$ of index $|\det X_J|$. On the other hand, it is easily seen that the sets $\{R_h(J) - \underline{\alpha} \mid \underline{\alpha} \in G_h^J\}$ form a partition of $\mathbb{R}^s$. This means that for every $\underline{y} \in A_h(\Gamma)$, each coset of G_h^J is uniquely represented in $b_h^J(\underline{y})$. In particular

$$(4.2) \quad |b_h^J(\underline{y})| = |\det X_J|, \quad \forall \underline{y} \in A_h(\Gamma).$$

(4.2) is actually a special case of [DM₂, Theorem 3.1] which will be used in its full generality in the next section.

THEOREM 4.1. Let $\underline{y} \in A_h(\Gamma)$. Then the operation of restricting elements of $\mathcal{S}_h(\Gamma)$ from $\mathbb{Z}_h^s$ to $b_h^\Gamma(\underline{y})$ is injective. In particular $\dim \mathcal{S}_h(\Gamma) \leq |b_h^\Gamma(\underline{y})|$.

PROOF. We prove the theorem by induction on $|\Gamma| \geq s$. Assume $|\Gamma| = s$ (and $\langle \Gamma \rangle = \mathbb{R}^s$). Let $u(\cdot) \in \mathcal{S}_h(\Gamma)$ and assume that $u|_{b_h^\Gamma(\underline{y})} \equiv 0$. Since $|\Gamma| = s$ we have $\langle \Gamma \setminus \gamma \rangle \neq \mathbb{R}^s$ for all $\gamma \in \Gamma$ and therefore

$$(4.3) \quad u(\cdot) \in \bigcap_{\gamma \in \Gamma} \ker \nabla_h^\gamma.$$

Using (4.3) it is easily seen that if $u(\underline{\alpha}^0) = 0$ for some $\underline{\alpha}^0 \in \mathbb{Z}_h^s$ then $u(\cdot)$ vanishes on the coset $\underline{\alpha}^0 + G_h^\Gamma$. Since $b_h^\Gamma(\underline{y})$ intersects every coset of G_h^Γ in $\mathbb{Z}_h^s$ we conclude $u \equiv 0$.

Assume $|\Gamma| > s$, $u \in \mathcal{S}(\Gamma)$, $u|_{b_h^\Gamma(\underline{y})} \equiv 0$. Let $\gamma \in \Gamma$. If $\langle \Gamma \setminus \gamma \rangle \neq \mathbb{R}^s$ then $\nabla_h^\gamma u \equiv 0$. If $\langle \Gamma \setminus \gamma \rangle = \mathbb{R}^s$ then we claim that

$$(4.4) \quad \nabla_h^\gamma u|_{b_h^{\Gamma \setminus \gamma}(\underline{y})} \equiv 0.$$

To see this assume $\underline{\alpha} \in b_h^{\Gamma \setminus \gamma}(\underline{y})$; then it follows from (1.4) and (1.12) that $\underline{\alpha}$, $\underline{\alpha} - h\underline{x}_\gamma \in b_h^\Gamma(\underline{y})$. Therefore $\nabla_h^\gamma u(\underline{\alpha}) = 0 - 0 = 0$.

By the induction hypothesis (4.4) implies $\nabla_h^\gamma u \equiv 0$. This implies that $\nabla_h^\gamma u = 0$ for all $\gamma \in \Gamma$. Let $J \in \mathbf{J}(\Gamma)$. Then $u|_{b_h^J(\underline{y})} \equiv 0$ because $b_h^J(\underline{y}) \subset b_h^\Gamma(\underline{y})$. Hence by the first part of the proof we conclude that $u \equiv 0$. $\square$

REMARK 4.1. It is clear that $\min\{|b_h^\Gamma(\underline{y})| \mid \underline{y} \in A_h(\Gamma)\}$ is independent of h . Therefore we see that the bound obtained in Theorem 4.1 to $\dim \mathcal{S}_h(\Gamma)$ is also independent of h .

REMARK 4.2. Comparison of Theorem 4.1 here with Theorem 3.3 of [DM₃] shows that Theorem 3.2 in [DM₃] is actually redundant.

We apply now Theorem 1.3 to show that $\mathcal{S}_h(\Gamma)$ and $\mathcal{H}(\Gamma)$ are *PE* spaces. Note that Theorem 1.3(b) was proved only with respect to sequences defined on $\mathbb{Z}^s$ rather than $\mathbb{Z}_h^s$. Yet, the extension to arbitrary h is trivial.

COROLLARY 4.1. $\mathcal{H}(\Gamma)$ and $\mathcal{S}_h(\Gamma)$ are finite dimensional *PE* spaces.

PROOF. In view of Theorem 1.3 and the obvious fact that $\mathcal{H}(\Gamma)$ is *D*-invariant and $\mathcal{S}_h(\Gamma)$ is shift invariant, it is enough to prove that these spaces are of finite dimension. For $\mathcal{S}_h(\Gamma)$ this was proved in Theorem 4.1.

To prove that $\mathcal{H}(\Gamma)$ is of finite dimension, let $N \in \mathbb{N}$ and assume that $\{f_j(\cdot)\}_{j=1}^N$ are linearly independent functions in $\mathcal{H}(\Gamma) \cap C^\infty(\mathbb{R}^s)$. Choose $h > 0$ small enough such that

$$u_j(\cdot) := f_j(\cdot)|_{\mathbb{Z}_h^s}, \quad j = 1, \dots, N,$$

are linearly independent on $\mathbb{Z}_h^s$. From (1.11) one easily deduces $u_j \in \mathcal{S}_h(\Gamma)$, $j = 1, \dots, N$. Thus since $\dim \mathcal{S}_h(\Gamma)$ is bounded uniformly in h , we see that $\mathcal{H}(\Gamma) \cap C^\infty(\mathbb{R}^s)$ is of finite dimension. Application of Lemma 3.2 completes the proof. $\square$

REMARK 4.3. Theorem 4.1 admits a continuous analog which shows in particular that $\dim \mathcal{H}(\Gamma) \leq |\mathbf{J}(\Gamma)|$ (see [DR]). That result can be used to give an alternative proof for Corollary 4.1, hence to derive Theorem 1.1 regardless of the restriction $X_\Gamma \subset \mathbb{Z}^s$.

Now, we use Corollary 4.1 to reveal the structure of $\mathcal{H}(\Gamma)$. For this purpose let $\Gamma_\underline{\theta}$ and $\Theta(\Gamma)$ be as in (2.2), (2.3) and denote for every $\underline{\theta} \in \Theta(\Gamma)$

$$(4.5) \quad X_{\underline{\theta}} = \{(\underline{x}, 0) \mid \underline{x} \in X_{\Gamma_\underline{\theta}}\},$$

i.e., $X_{\underline{\theta}}$ is the defining set obtained from $\Gamma_\underline{\theta}$ when replacing $\lambda_{\Gamma_\underline{\theta}}$ by $\underline{0}$.

THEOREM 4.2. The space $\mathcal{H}(\Gamma)$ admits the following direct sum decomposition

$$(4.6) \quad \mathcal{H}(\Gamma) = \bigoplus_{\underline{\theta} \in \Theta(\Gamma)} \mathcal{H}(\Gamma_\underline{\theta}) = \bigoplus_{\underline{\theta} \in \Theta(\Gamma)} e_{\underline{\theta}}(\cdot) \mathcal{H}(X_{\underline{\theta}}),$$

where each $\mathcal{H}(X_{\underline{\theta}})$ is a nontrivial polynomial space. In particular

$$(4.7) \quad e_{\underline{\theta}}(\cdot) \in \mathcal{H}(\Gamma) \quad \text{if and only if} \quad \underline{\theta} \in \Theta(\Gamma).$$

PROOF. By Corollary 4.1 it is enough to identify in $\mathcal{H}(\Gamma)$ the functions of the form $e_{\underline{\theta}}(\cdot)p(\cdot)$, $p \in \pi$, $\underline{\theta} \in \mathbb{C}^s$. We start by proving that $e_{\underline{\theta}}(\cdot)p(\cdot) \in \mathcal{H}(\Gamma)$ only if $\underline{\theta} \in \Theta(\Gamma)$. Given such a function we see that for $\gamma \in \Gamma_{\underline{\theta}}$

$$D^{\gamma}(e_{\underline{\theta}}(\cdot)p(\cdot)) = e_{\underline{\theta}}(\cdot)D_{\underline{x}_{\gamma}}p(\cdot),$$

while for $\gamma \in \Gamma \setminus \Gamma_{\underline{\theta}}$

$$D^{\gamma}(e_{\underline{\theta}}(\cdot)p(\cdot)) = e_{\underline{\theta}}(\cdot)q(\cdot), \quad \text{where } \deg q = \deg p.$$

Thus for $K \subset \Gamma$

$$(4.8) \quad D^K(e_{\underline{\theta}}(\cdot)p(\cdot)) \equiv 0 \Leftrightarrow D^{K \cap \Gamma_{\underline{\theta}}}(e_{\underline{\theta}}(\cdot)p(\cdot)) \equiv 0 \Leftrightarrow D^{X_{\Gamma_{\underline{\theta}} \cap K}}p(\cdot) \equiv 0.$$

If $\underline{\theta} \notin \Theta(\Gamma)$ then $\langle \Gamma_{\underline{\theta}} \rangle \neq \mathbb{R}^s$ and hence $\Gamma \setminus \Gamma_{\underline{\theta}} \in \mathcal{H}(\Gamma)$. On the other hand, (4.8) shows that $D^{\Gamma \setminus \Gamma_{\underline{\theta}}}(e_{\underline{\theta}}(\cdot)p(\cdot)) \neq 0$ because $(\Gamma \setminus \Gamma_{\underline{\theta}}) \cap \Gamma_{\underline{\theta}} = \emptyset$. Consequently $e_{\underline{\theta}}(\cdot)p(\cdot) \notin \mathcal{H}(\Gamma)$. Combining this result with Proposition 2.1 verifies (4.7), and since $\Theta(X_{\underline{\theta}}) = \{0\}$ for every $\underline{\theta} \in \Theta(\Gamma)$, we also conclude that $\mathcal{H}(X_{\underline{\theta}})$ contains only polynomials. Finally, (4.6) easily follows from (4.8) and the observation that

$$\mathcal{H}(\Gamma_{\underline{\theta}}) = \{K \cap \Gamma_{\underline{\theta}} \mid K \in \mathcal{H}(\Gamma)\}. \quad \square$$

We now turn to the analysis of $\mathcal{S}_h(\Gamma)$. Here for $\underline{\theta} \in \mathbb{C}^s$ we define

$$(4.9) \quad \tilde{\Gamma}_{\underline{\theta},h} = \{\gamma \in \Gamma \mid e^{h(\lambda_{\gamma} - \underline{x}_{\gamma} \cdot \underline{\theta})} = 1\},$$

and introduce the set of “discrete nodes”

$$(4.10) \quad \tilde{\Theta}_h(\Gamma) = \{\underline{\theta} \in \mathbb{C}^s \mid \langle \tilde{\Gamma}_{\underline{\theta},h} \rangle = \mathbb{R}^s, \text{Im } \underline{\theta} \in [0, 2\pi h^{-1})^s\}.$$

To analyze $\mathcal{S}_h(\Gamma)$ we first prove the following result.

LEMMA 4.1. *Let $K \subset \Gamma$, $p \in \pi$. Then*

$$\nabla_h^{X_K} p \equiv 0 \Rightarrow D^{X_K} p \equiv 0.$$

PROOF. Given a compactly supported $\phi \in \mathcal{D}'(\mathbb{R}^s)$ and $q \in \pi$ it can be shown that

$$\phi * q = \hat{\phi}(0)q(\underline{x}) + \text{a polynomial of lower degree than } q(\underline{x}).$$

In particular $\phi * q \equiv 0$ and $\hat{\phi}(0) \neq 0$ imply $q \equiv 0$. Now assuming $\nabla_h^{X_K} p \equiv 0$ we get from (1.11) that $B_h(X_K) * D^{X_K} p \equiv 0$, and since $\hat{B}_h(X_K|0) \neq 0$ then $D^{X_K} p \equiv 0$. $\square$

Now, we combine Corollary 4.1 together with Lemma 4.1 and Proposition 2.1 to conclude

THEOREM 4.3.

$$(4.11) \quad \mathcal{S}_h(\Gamma) = \bigoplus_{\underline{\theta} \in \tilde{\Theta}_h(\Gamma)} e_{\underline{\theta}}(\cdot) \mathcal{H}(\tilde{X}_{\underline{\theta},h}),$$

where

$$\tilde{X}_{\underline{\theta},h} = \{(\underline{x}_{\gamma}, 0) \mid \gamma \in \tilde{\Gamma}_{\underline{\theta},h}\}.$$

REMARK 4.4. By the convention stated in the introduction only restrictions to $\mathbb{Z}_h^s$ are considered in (4.11).

PROOF. The sum $\bigoplus_{\underline{\theta} \in \tilde{\Theta}_h(\Gamma)} e_{\underline{\theta}}(\cdot) \mathcal{H}(\tilde{X}_{\underline{\theta},h})$ is indeed direct. This follows from (4.10), Proposition 3.1 and the fact that each $\mathcal{H}(\tilde{X}_{\underline{\theta},h})$ contains only polynomials.

Now, for $\underline{\theta} \in \mathbb{C}^s$, $\gamma \in \tilde{\Gamma}_{\underline{\theta},h}$ and $p \in \pi$

$$\nabla_h^\gamma(e_{\underline{\theta}}(\cdot)p(\cdot)) = e_{\underline{\theta}}(\cdot)\nabla_h^{\underline{x}_\gamma}p(\cdot),$$

while for $\gamma \notin \tilde{\Gamma}_{\underline{\theta},h}$

$$\nabla_h^\gamma(e_{\underline{\theta}}(\cdot)p(\cdot)) = e_{\underline{\theta}}(\cdot)q(\cdot) \quad \text{where } \deg p = \deg q.$$

Utilizing Lemma 4.1 we see that our claim is obtained by the same arguments used in the proof of Theorem 4.2. $\square$

Usually the set of discrete nodes $\tilde{\Theta}_h(\Gamma)$ is richer than that of nodes $\Theta(\Gamma)$. On the other hand, if $\underline{\theta} \in \tilde{\Theta}_h(\Gamma) \cap \Theta(\Gamma)$ we may have $\Gamma_{\underline{\theta}} \subsetneq \tilde{\Gamma}_{\underline{\theta},h}$. Yet, this last case is not “stable” and can be avoided by a refinement of the mesh.

PROPOSITION 4.1. *For every Γ there exists a positive number h_0 , which depends only on X_Γ and $\|\underline{\lambda}_\Gamma\|_\infty$, such that for every $h < h_0$*

(a) $\Theta(\Gamma) \subset \tilde{\Theta}_h(\Gamma)$, and

(b) $\Gamma_{\underline{\theta}} = \tilde{\Gamma}_{\underline{\theta},h}, \forall \underline{\theta} \in \Theta(\Gamma)$.

Moreover, if $\underline{\lambda}_\Gamma \subset \mathbb{R}$ then (a), (b) hold for every $h > 0$.

PROOF. It is easy to see that there exists $c_1 > 0$ which depends only on X_Γ such that

$$(4.12) \quad \|\underline{\theta}\|_\infty \leq c_1 \|\underline{\lambda}_\Gamma\|_\infty, \quad \forall \underline{\theta} \in \Theta(\Gamma).$$

So, in view of (4.10), the choice $h < 2\pi(c_1\|\underline{\lambda}_\Gamma\|_\infty)^{-1}$ gives (a). Given $\underline{\theta} \in \mathbb{C}^s$ we see from (4.9) that

$$\gamma \in \tilde{\Gamma}_{\underline{\theta},h} \Leftrightarrow h(\lambda_\gamma - \underline{x}_\gamma \cdot \underline{\theta}) \in 2\pi i\mathbb{Z}^s.$$

But (4.12) implies the existence of $c_2 \geq c_1$ (dependent only on X_Γ) such that $|\lambda_\gamma - \underline{x}_\gamma \cdot \underline{\theta}| \leq c_2 \|\underline{\lambda}_\Gamma\|_\infty$ for all $\gamma \in \Gamma$, $\underline{\theta} \in \Theta(\Gamma)$. Thus given $\underline{\theta} \in \Theta(\Gamma)$ and $\gamma \in \Gamma \setminus \Gamma_{\underline{\theta}}$ we have for $h < h_0 = 2\pi(c_2\|\underline{\lambda}_\Gamma\|_\infty)^{-1}$

$$0 < |h(\lambda_\gamma - \underline{x}_\gamma \cdot \underline{\theta})| < hc_2\|\underline{\lambda}_\Gamma\|_\infty < 2\pi,$$

which shows that $\gamma \notin \tilde{\Gamma}_{\underline{\theta},h}$. Thus $\Gamma_{\underline{\theta}} \supset \tilde{\Gamma}_{\underline{\theta},h}$ whereas the opposite inclusion is trivial.

Finally, note that if $\underline{\lambda}_\Gamma \subset \mathbb{R}$ then clearly $\Theta(\Gamma) \subset \mathbb{R}^s$ so that for every $h > 0$,

$$\gamma \in \tilde{\Gamma}_{\underline{\theta},h} \Leftrightarrow h(\lambda_\gamma - \underline{x}_\gamma \cdot \underline{\theta}) \in 2\pi i\mathbb{Z}^s \Leftrightarrow \lambda_\gamma - \underline{x}_\gamma \cdot \underline{\theta} = 0 \Leftrightarrow \gamma \in \Gamma_{\underline{\theta}}$$

which proves (b), while (a) in this case is trivial. $\square$

Note that Proposition 4.1 says that for small enough h , $\mathcal{H}(\Gamma)$ is embedded in $\mathcal{S}_h(\Gamma)$. Note also that the proof of this proposition shows that $\tilde{\Theta}_h(\Gamma) - \Theta(\Gamma)$ tends to infinity as h tends to zero.

Comparing (4.9) and (1.2) we see that for $\underline{\theta} \in \Theta(\Gamma)$

$$(4.13) \quad \tilde{\Gamma}_{\underline{\theta},h} = \Gamma_{\underline{\theta}} \Leftrightarrow \hat{B}_h(\Gamma| - i\underline{\theta}) \neq 0.$$

Therefore Proposition 4.1 implies the following corollary.

COROLLARY 4.2. *For every Γ there exists a positive h_0 , which depends only on X_Γ and $\|\underline{\lambda}_\Gamma\|_\infty$ such that for every $h < h_0$*

$$(4.14) \quad \hat{B}_h(\Gamma| - i\underline{\theta}) \neq 0, \quad \forall \underline{\theta} \in \Theta(\Gamma).$$

In case $\underline{\lambda}_\Gamma \subset \mathbb{R}$, (4.14) holds for every h .

5. Proof of Theorem 1.1. Here we complete the proof of Theorem 1.1 and establish several additional properties of $\mathcal{H}(\Gamma)$ and $\mathcal{S}_h(\Gamma)$.

First we derive the following quantitative result about the set $\tilde{\Theta}_h(\Gamma)$ of discrete nodes:

LEMMA 5.1. *For any $J \in \mathbf{J}(\Gamma)$*

$$|\tilde{\Theta}_h(J)| = |\det X_J|.$$

PROOF. Fix $\underline{\theta}^0 \in \tilde{\Theta}_h(J)$ and let $\underline{\theta} \in \mathbb{C}^s$ be arbitrary. Then by (4.9), (4.10), $\underline{\theta} \in \tilde{\Theta}_h(J)$ if and only if $e^{h(\underline{\theta} - \underline{\theta}^0) \cdot \underline{x}_\gamma} = 1$, $\forall \gamma \in J$. This condition is equivalent to $e_{\underline{\theta} - \underline{\theta}^0}(\cdot)$ being a complex homomorphism of $\mathbb{Z}_h^s$ which is constant on the cosets of G_h^J in $\mathbb{Z}_h^s$. Each such homomorphism can be factored as follows

$$\begin{array}{ccc} \mathbb{Z}_h^s & \xrightarrow{q} & \mathbb{Z}_h^s / G_h^J \\ e_{\underline{\theta} - \underline{\theta}^0} \searrow & & \swarrow \chi \\ & \mathbb{C} & \end{array}$$

where q is the quotient map and χ is a character of the finite abelian group $\mathbb{Z}_h^s / G_h^J$. Thus we established a one-to-one correspondence between the characters of this last group and the elements of $\tilde{\Theta}_h(J)$ and our claim follows from the fact that

$$|\mathbb{Z}_h^s / G_h^J| = |\det X_J|. \quad \square$$

In the proof of Theorem 1.1 we will also use the following combinatorial result which is due to Dahmen and Micchelli.

PROPOSITION 5.1 [DM₂, THEOREM 3.1]. *For every $h > 0$ and every $\underline{y} \in A_h(\Gamma)$*

$$|b_h^\Gamma(\underline{y})| = \sum_{J \in \mathbf{J}(\Gamma)} |\det X_J|.$$

PROOF OF THEOREM 1.1. Part (a) was established in Corollary 4.1. Application of Theorem 2.1 to each of the components in (4.11) yields

$$(5.1) \quad \dim \mathcal{S}_h(\Gamma) \geq \sum_{\underline{\theta} \in \tilde{\Theta}_h(\Gamma)} |\mathbf{J}(\tilde{\Gamma}_{\underline{\theta}, h})|,$$

with equality if and only if

$$(5.2) \quad \dim \mathcal{H}(\tilde{\Gamma}_{\underline{\theta}, h}) = |\mathbf{J}(\tilde{\Gamma}_{\underline{\theta}, h})|, \quad \forall \underline{\theta} \in \tilde{\Theta}_h(\Gamma).$$

By Lemma 5.1 we know that each $J \in \mathbf{J}(\Gamma)$ lies exactly in $|\det X_J|$ sets from $\{\tilde{\Gamma}_{\underline{\theta}, h} | \underline{\theta} \in \tilde{\Theta}_h(\Gamma)\}$, so

$$\sum_{\underline{\theta} \in \tilde{\Theta}_h(\Gamma)} |\mathbf{J}(\tilde{\Gamma}_{\underline{\theta}, h})| = \sum_{J \in \mathbf{J}(\Gamma)} |\det X_J|.$$

Hence (5.1) implies

$$\dim \mathcal{S}_h(\Gamma) \geq \sum_{J \in \mathbf{J}(\Gamma)} |\det X_J|,$$

whereas Theorem 4.1 says that

$$\dim \mathcal{S}_h(\Gamma) \leq |b_h^\Gamma(\underline{y})|, \quad \forall \underline{y} \in A_h(\Gamma).$$

Therefore, application of Proposition 5.1 completes the proof of (c), which in view of Theorem 4.1 also gives (d).

To prove (b) note that we have just shown that equality holds in (5.1); hence (5.2) is valid for every $h > 0$. Thus by appealing to Proposition 4.1 we obtain

$$(5.3) \quad \dim \mathcal{H}(\Gamma_\theta) = |\mathbf{J}(\Gamma_\theta)|, \quad \forall \theta \in \Theta(\Gamma),$$

and since every $J \in \mathbf{J}(\Gamma)$ is contained exactly in one set from $\{\Gamma_\theta \mid \theta \in \Theta(\Gamma)\}$, (b) follows from Theorem 4.2. $\square$

Combining Theorem 1.1(b) and Proposition 2.1 one obtains the following result which could also be verified directly:

COROLLARY 5.1. *Assume Γ is simple. Then*

$$\mathcal{H}(\Gamma) = \bigoplus_{\theta \in \Theta(\Gamma)} e_\theta(\cdot).$$

As a by-product we also derive

THEOREM 5.1. *Let Γ be a defining set and fix $h > 0$. Then the following conditions are equivalent:*

(a) *To every $u \in \mathcal{S}_h(\Gamma)$ there corresponds a unique $f \in \mathcal{H}(\Gamma)$ which interpolates $u(\cdot)$ (on $\mathbb{Z}_h^s$).*

(b)

$$(5.4) \quad |\det X_J| = 1, \quad \forall J \in \mathbf{J}(\Gamma),$$

$$(5.5) \quad \underline{\theta}^k - \underline{\theta}^j \notin h^{-1}2\pi i\mathbb{Z}^s \setminus \{0\}, \quad \forall \underline{\theta}^k, \underline{\theta}^j \in \Theta(\Gamma).$$

PROOF. Comparing parts (b) and (c) of Theorem 1.1 the necessity of (5.4) is clear. Assuming (5.4) we know from Theorem 1.1 that $\dim \mathcal{S}_h(\Gamma) = \dim \mathcal{H}(\Gamma)$ whereas by (1.11) $\mathcal{H}(\Gamma)|_{\mathbb{Z}_h^s} \subset \mathcal{S}_h(\Gamma)$. So to establish (a) it is necessary and sufficient that restriction to $\mathbb{Z}_h^s$ will be one to one on $\mathcal{H}(\Gamma)$, which by Theorem 4.2 and Proposition 3.1 is equivalent to (5.5). $\square$

Part (d) of Theorem 1.1 can be restated as follows:

COROLLARY 5.2. *For every $\underline{\alpha} \in \mathbb{R}^s$ denote by $\delta_{\underline{\alpha}}$ the linear functional of evaluation at $\underline{\alpha}$. Then for every $\underline{x} \in A_h(\Gamma)$ the set $\{\delta_{\underline{\alpha}}\}_{\underline{\alpha} \in b_h^\Gamma(\underline{x})}$ forms a basis to the dual of $\mathcal{S}_h(\Gamma)$. Namely every linear functional ρ on $\mathcal{S}_h(\Gamma)$ has a unique representation of the form*

$$(5.6) \quad \rho(u) = \sum_{\underline{\alpha} \in b_h^\Gamma(\underline{x})} c_{\underline{\alpha}} u(\underline{\alpha}), \quad \forall u \in \mathcal{S}_h(\Gamma).$$

Now, when (5.5) holds we know from Proposition 3.1 that $\mathcal{H}(\Gamma)$ is embedded in $\mathcal{S}_h(\Gamma)$ and thus Corollary 5.2 yields the following.

COROLLARY 5.3. *Let $\underline{x} \in A_h(\Gamma)$ and assume (5.5) holds. Then every linear functional ρ on $\mathcal{H}(\Gamma)$ has a representation as in (5.6). This representation is unique if and only if (5.4) holds.*

Finally we are interested in the case of “ h -discrete simplicity”, namely the case where $\mathcal{S}_h(\Gamma)$ is spanned by pure exponentials. In general such requirement is stronger than the usual simplicity. Yet we have

THEOREM 5.2. *To every simple defining set Γ there corresponds a positive number h_0 , which depends only on X_Γ and $\|\underline{\lambda}_\Gamma\|_\infty$, such that for $h < h_0$*

$$(5.7) \quad \mathcal{S}_h(\Gamma) = \bigoplus_{\underline{\theta} \in \tilde{\Theta}_h(\Gamma)} e_{\underline{\theta}}(\cdot).$$

Moreover, if $\underline{\lambda}_\Gamma \subset \mathbb{R}$ then (5.7) holds for every $h > 0$.

PROOF. If Γ does not satisfy (5.7) then there exists $\underline{\theta} \in \tilde{\Theta}_h(\Gamma)$ such that $|\tilde{\Gamma}_{\underline{\theta},h}| \geq s+1$ and hence contains at least two bases J_1, J_2 from $\mathbf{J}(\Gamma)$. Let $\underline{\theta}^j$, $j=1,2$, be the only node of $\Theta(J_j)$, $j=1,2$. By the proof of Lemma 5.1 we know that

$$\underline{\theta} = \underline{\theta}^1 + i\underline{\nu}^1 = \underline{\theta}^2 + i\underline{\nu}^2$$

where $\underline{\nu}^1, \underline{\nu}^2 \in \mathbb{R}^s$ and

$$e^{hi\underline{\nu}^j \cdot \underline{x}_\gamma} = 1, \quad \forall \gamma \in J_j, \quad j=1,2.$$

In particular we have $hm(\underline{\nu}^1 - \underline{\nu}^2) \in 2\pi\mathbb{Z}^s$ where m is the common multiple of $\{|\det X_J| \mid J \in \mathbf{J}(\Gamma)\}$.

Now, if $\underline{\lambda}_\Gamma \subset \mathbb{R}$ then also $\Theta(\Gamma) \subset \mathbb{R}^s$, so the equality $\underline{\theta}^1 + i\underline{\nu}^1 = \underline{\theta}^2 + i\underline{\nu}^2$ implies $\underline{\theta}^1 = \underline{\theta}^2$ which is impossible since Γ is simple. Consequently (5.7) holds in this case for every $h > 0$.

For general $\underline{\lambda}_\Gamma$ we use the fact that there exists $c > 0$ independent of $\underline{\lambda}_\Gamma$ such that $\|\underline{\theta}\|_\infty \leq c\|\underline{\lambda}_\Gamma\|_\infty$ for all $\underline{\theta} \in \Theta(\Gamma)$. Choosing $t_0 > \|\underline{\lambda}_\Gamma\|_\infty$ we take $h_0 = 1/2cmt_0$. Then for $h < h_0$

$$\|hmi(\underline{\nu}^1 - \underline{\nu}^2)\|_\infty = hm\|\underline{\theta}^1 - \underline{\theta}^2\|_\infty < \frac{1}{2cmt_0} 2mct_0 = 1,$$

forcing $\underline{\nu}^1 = \underline{\nu}^2$ and therefore $\underline{\theta}^1 = \underline{\theta}^2$ which again contradicts the simplicity of Γ . Hence every $\tilde{\Gamma}_{\underline{\theta},h}$, $\underline{\theta} \in \tilde{\Theta}_h(\Gamma)$, contains exactly s elements and (5.7) follows. $\square$

6. Exponential box splines and their related spaces. In this section we apply some of the results of the previous sections to exponential box splines.

Let Γ and $h > 0$ be given. For a sequence $u(\cdot)$ defined on $\mathbb{Z}_h^s$ we denote

$$(6.1) \quad B_h(\Gamma) *_{\mathbf{h}} u := \sum_{\underline{\alpha} \in \mathbb{Z}_h^s} u(\underline{\alpha}) B_h(\Gamma | \cdot - \underline{\alpha}).$$

The range of the map $B_h(\Gamma) *_{\mathbf{h}}$ is denoted by $\mathbf{S}_h(\Gamma)$ while $\mathbf{H}_h(\Gamma)$ stands for the subspace of all entire functions in $\mathbf{S}_h(\Gamma)$. Our first claim is the following.

PROPOSITION 6.1. *For any defining set Γ*

$$(6.2) \quad \mathbf{H}_h(\Gamma) \subset \mathcal{H}(\Gamma).$$

PROOF. Assume that $f \in \mathbf{H}_h(\Gamma)$. Then $f \in B_h(\Gamma) *_{\hbar} u$ for some sequence $u(\cdot)$ defined on $\mathbb{Z}_h^s$. Let $K \in \mathcal{H}(\Gamma)$. Then by (1.15)

$$D^K f = D^K (B_h(\Gamma) *_{\hbar} u) = (D^K B_h(\Gamma)) *_{\hbar} u = \nabla_h^K [B_h(\Gamma \setminus K) *_{\hbar} u].$$

Now, $K \in \mathcal{H}(\Gamma)$ means $\langle \Gamma \setminus K \rangle \neq \mathbf{R}^s$. Thus (1.4) shows that $\text{supp } B_h(\Gamma \setminus K)$ lies on a hyperplane of $\mathbf{R}^s$. We conclude that $D^K f$ is an entire function whose support is contained in a countable union of hyperplanes. Consequently $D^K f \equiv 0$, which implies $\circ$

$$f \in \bigcap_{K \in \mathcal{H}(\Gamma)} \ker D^K = \mathcal{H}(\Gamma). \quad \square$$

In certain cases $\mathbf{H}_h(\Gamma)$ is a proper subspace of $\mathcal{H}(\Gamma)$ (cf. [R₂, Ex. 4.1]). Conditions for equality in (6.2) will be derived later.

The significance of $\mathcal{S}_h(\Gamma)$ in the present analysis is due to the following theorem:

THEOREM 6.1. *The map $B_h(\Gamma) *_{\hbar}$ carries $\mathcal{S}_h(\Gamma)$ onto $\mathbf{H}_h(\Gamma)$.*

PROOF. We first show that $B_h(\Gamma) *_{\hbar}$ maps $\mathcal{S}_h(\Gamma)$ into $\mathbf{H}_h(\Gamma)$. Let $u(\cdot) \in \mathcal{S}_h(\Gamma)$ and denote $f_u = B_h(\Gamma) *_{\hbar} u$. For each $K \in \mathcal{H}(\Gamma)$ we have by (1.15)

$$(6.3) \quad D^K f_u = [D^K B_h(\Gamma)] *_{\hbar} u = [\nabla_h^K B_h(\Gamma \setminus K)] *_{\hbar} u = B_h(\Gamma \setminus K) *_{\hbar} \nabla_h^K u = 0.$$

Hence $f_u \in \mathcal{H}(\Gamma)$. By Theorem 4.2 $\mathcal{H}(\Gamma)$ contains only entire functions. Thus by definition $f_u \in \mathbf{H}_h(\Gamma)$.

We now prove that this map is onto. Let $f = B_h(\Gamma) *_{\hbar} u_1 \in \mathbf{H}_h(\Gamma)$ be arbitrary, where $u_1(\cdot) \in \mathcal{S}_h(\Gamma)$. Choose any $\underline{y} \in A_h(\Gamma)$. By Theorem 1.1(d) there exists an element $u_2 \in \mathcal{S}_h(\Gamma)$ satisfying

$$(6.4) \quad u_2|_{b_h^\Gamma(\underline{y})} = u_1|_{b_h^\Gamma(\underline{y})}.$$

Denote $\tilde{f} = B_h(\Gamma) *_{\hbar} (u_2 - u_1)$. From the first part of the proof $B_h(\Gamma) *_{\hbar} u_2$ is entire and therefore $\tilde{f}$ is entire. Let A be the $\Gamma - h$ cell containing $\underline{y}$. Then for every $\underline{x} \in A$,

$$\tilde{f}(\underline{x}) = \sum_{\underline{\alpha} \in \mathbb{Z}_h^s} (u_1 - u_2)(\underline{\alpha}) B_h(\Gamma | \underline{x} - \underline{\alpha}) = \sum_{\underline{\alpha} \in b_h^\Gamma(\underline{y})} (u_1 - u_2)(\underline{\alpha}) B_h(\Gamma | \underline{x} - \underline{\alpha}),$$

because $B_h(\Gamma | \underline{x} - \underline{\alpha}) = 0$ whenever $\underline{x} \in A$ and $\underline{\alpha} \notin b_h^\Gamma(\underline{y})$. Taking into account (6.4) we obtain $\tilde{f}|_A \equiv 0$. Since $\tilde{f}$ is entire the last equality readily implies $\tilde{f} \equiv 0$ on $\mathbf{R}^s$. Thus $B_h(\Gamma) *_{\hbar} u_2 = f$. $\square$

PROOF OF THEOREM 1.4. The theorem follows from Proposition 6.1 and Theorem 6.1 above. $\square$

In the next two results we shall make use of the following lemma:

LEMMA 6.1. *Given $\theta \in \Theta(\Gamma)$, consider the following two conditions:*

(a) *There exists $\underline{\theta}^0 \neq \underline{\theta}$ in $\Theta(\Gamma)$ such that $h(\underline{\theta}^0 - \underline{\theta}) \in 2\pi i \mathbb{Z}^s$.*

(b) $\hat{B}_h(\Gamma | -i\underline{\theta}) = 0$.

Then (a) always implies (b). If in addition we assume

$$(6.5) \quad |\det X_J| = 1, \quad \forall J \in \mathbf{J}(\Gamma),$$

then (b) implies (a) as well.

PROOF. Assume (a). Since $\underline{\theta}^0 - \underline{\theta} \neq 0$ and $\langle \Gamma_{\underline{\theta}^0} \rangle = \mathbb{R}^s$, there exists $\gamma \in \Gamma_{\underline{\theta}^0}$ such that $\underline{x}_\gamma \cdot (\underline{\theta} - \underline{\theta}^0) \neq 0$. But $\underline{x}_\gamma \in \mathbb{Z}^s$. Therefore $h(\lambda_\gamma - \underline{x}_\gamma \cdot \underline{\theta}) = h\underline{x}_\gamma \cdot (\underline{\theta}^0 - \underline{\theta}) \in 2\pi i\mathbb{Z} \setminus \{0\}$. This implies that $\int_0^h e^{(\lambda_\gamma - \underline{x}_\gamma \cdot \underline{\theta})t} dt = 0$, which in view of (1.2) gives $\hat{B}_h(\Gamma| - i\underline{\theta}) = 0$.

For the converse assume $\hat{B}_h(\Gamma| - i\underline{\theta}) = 0$. Then by (1.2) there exists $\gamma_0 \in \Gamma$ such that

$$(6.6) \quad h(\lambda_{\gamma_0} - \underline{x}_{\gamma_0} \cdot \underline{\theta}) \in 2\pi i\mathbb{Z} \setminus \{0\}.$$

Complete γ_0 to a basis $J_0 \in \mathbf{J}(\Gamma)$ by elements of $\Gamma_{\underline{\theta}}$ and let $\underline{\theta}^0$ be the node in $\Theta(\Gamma)$ which satisfies $\underline{x}_\gamma \cdot \underline{\theta}^0 - \lambda_\gamma = 0$ for every $\gamma \in J_0$. Given $\gamma \in J_0 \setminus \gamma_0$ we know that $\gamma \in \Gamma_{\underline{\theta}} \cap \Gamma_{\underline{\theta}^0}$ and therefore

$$h(\underline{x}_\gamma \cdot (\underline{\theta}^0 - \underline{\theta})) = h(\lambda_\gamma - \lambda_\gamma) = 0.$$

From (6.6) and the last equation we conclude that $h(\underline{\theta}^0 - \underline{\theta})/2\pi i$ coincides with an integer multiple of one of the columns of $(X_{J_0})^{-1}$, which, since $|\det X_{J_0}| = 1$, contains only integer entries. This proves (a). $\square$

Given $f \in \mathcal{H}(\Gamma)$ we know from (1.11) that $f|_{\mathbf{z}_h^s} \in \mathcal{S}_h(\Gamma)$. Hence Theorem 6.1 yields

$$(6.7) \quad B_h(\Gamma) *'_h (f|_{\mathbf{z}_h^s}) \in \mathbf{H}_h(\Gamma), \quad \forall f \in \mathcal{H}(\Gamma).$$

The next result gives a necessary and sufficient condition for $B_h(\Gamma) *'_h$ being injective on $\mathcal{H}(\Gamma)|_{\mathbf{z}_h^s}$.

THEOREM 6.2. *Define a map $B_h(\Gamma) *'_h : C(\mathbb{R}^s)$ by*

$$B_h(\Gamma) *'_h : f \rightarrow B_h(\Gamma) *'_h (f|_{\mathbf{z}_h^s}).$$

Let $\Gamma_1 \subset \Gamma$ and assume $\langle \Gamma_1 \rangle = \mathbb{R}^s$. Then

- (a) $\mathcal{H}(\Gamma_1)$ is an invariant subspace of $B_h(\Gamma) *'_h$.
- (b) $B_h(\Gamma) *'_h$ is an automorphism of $\mathcal{H}(\Gamma_1)$ if and only if

$$(6.8) \quad \hat{B}_h(\Gamma| - i\underline{\theta}) \neq 0, \quad \forall \underline{\theta} \in \Theta(\Gamma_1).$$

PROOF. Let $f \in \mathcal{H}(\Gamma_1)$ and let $K \in \mathcal{H}(\Gamma_1)$. Then $D^K f \equiv 0$ and therefore by (1.11) $\nabla_h^K f \equiv 0$. Hence (as in the proof of the first part of Theorem 6.1)

$$D^K (B_h(\Gamma) *'_h f) = B_h(\Gamma \setminus K) *'_h \nabla_h^K f \equiv 0.$$

This proves (a).

Now denote

$$\text{Ker}_h(\Gamma_1) = \{f \in \mathcal{H}(\Gamma_1) \mid B_h(\Gamma) *'_h f = 0\},$$

and assume (6.8) holds. By Lemma 6.1 we know that

$$h(\underline{\theta}^1 - \underline{\theta}^2) \notin 2\pi i\mathbb{Z}^s \setminus \{0\}, \quad \forall \underline{\theta}^1, \underline{\theta}^2 \in \Theta(\Gamma_1).$$

Hence we can combine Theorem 4.2, Proposition 3.1, Corollary 3.1 and the obvious fact that $\text{Ker}_h(\Gamma_1)|_{\mathbf{z}_h^s}$ is shift-invariant to conclude that if $\text{Ker}_h(\Gamma_1)$ is not trivial

then it must contain an exponential $e_{\underline{\theta}^0}$, $\underline{\theta}^0 \in \Theta(\Gamma_1)$. Now we appeal to [R₁, Corollary 5.1] which states that

$$(6.9) \quad B_h(\Gamma) *'_h e_{\underline{\theta}} = h^{-s} \hat{B}_h(\Gamma | -i\underline{\theta}) e_{\underline{\theta}}, \quad \forall \underline{\theta} \in \Theta(\Gamma),$$

to conclude that $\hat{B}_h(\Gamma | -i\underline{\theta}^0) = 0$, which contradicts (6.8). Consequently $B_h(\Gamma) *'_h$ is injective on $\mathcal{H}(\Gamma_1)$ and therefore by the first part of the proof induces an automorphism on $\mathcal{H}(\Gamma_1)$.

For the converse assume (6.8) is false. Then $\hat{B}_h(\Gamma | -i\underline{\theta}) = 0$ for some $\underline{\theta} \in \Theta(\Gamma_1)$. Taking (6.9) into account we see that $e_{\underline{\theta}} \in \text{Ker}_h(\Gamma)$. By Theorem 4.2 $e_{\underline{\theta}} \in \mathcal{H}(\Gamma_1)$ and hence $B_h(\Gamma) *'_h$ is not injective on $\mathcal{H}(\Gamma_1)$. $\square$

COROLLARY 6.1. *Assume that*

$$(6.10) \quad \hat{B}_h(\Gamma | -i\underline{\theta}) \neq 0, \quad \forall \underline{\theta} \in \Theta(\Gamma).$$

Then

$$(6.11) \quad \mathbf{H}_h(\Gamma) = \mathcal{H}(\Gamma).$$

Application of Theorem 1.1(b) gives

COROLLARY 6.2. *Assume (6.10) holds. Then $\dim \mathbf{H}_h(\Gamma) = |\mathbf{J}(\Gamma)|$.*

For polynomial box splines (namely in the case when $\underline{\lambda}_\Gamma = \underline{0}$) $\Theta(\Gamma) = \{\underline{0}\}$ and $\hat{B}_h(\Gamma | \underline{0}) = h^{|\Gamma|} \neq 0$. Therefore in this case (6.11) always holds. This result is due to de Boor and Höllig, [BH], and proved by a different method in [DM₁]. The fact that (6.11) is valid for *polynomial* box splines was used in [R₂] to establish a similar result to that of Corollary 6.1. We also mention that (6.10) is not a *necessary* condition for (6.11) (cf. [R₂, Ex. 4.2]).

Although Corollary 6.1 is valid for every choice of Γ , a relatively simpler proof exists for the choice $\underline{\lambda}_\Gamma \subset \mathbb{R}$. By Corollary 4.2 we know that (6.10) is always valid in this case. Hence

COROLLARY 6.1*. *Assume $\underline{\lambda}_\Gamma \subset \mathbb{R}$. Then $\mathbf{H}_h(\Gamma) = \mathcal{H}(\Gamma)$.*

PROOF. By Theorem 6.2(a) we know that for each $\underline{\theta} \in \Theta(\Gamma)$, $B_h(\Gamma) *'_h$ maps $\mathcal{H}(\Gamma_{\underline{\theta}})$ into itself. Since $\underline{\lambda}_\Gamma \subset \mathbb{R}$ it follows that $B_h(\Gamma)$ is *positive* and therefore (as mentioned in [BD]) $B_h(\Gamma) *'_h$ is injective on polynomials. But by Theorem 4.2 $\mathcal{H}(\Gamma_{\underline{\theta}}) = e_{\underline{\theta}} \mathcal{H}(X_{\underline{\theta}})$ where $\mathcal{H}(X_{\underline{\theta}})$ is a finite dimensional polynomial space. Now the desired result readily follows. $\square$

Finally we derive necessary and sufficient conditions for $\mathcal{S}_h(\Gamma)$ being isomorphic to $\mathbf{H}_h(\Gamma)$

THEOREM 6.3. *The following conditions are equivalent:*

- (a) $B_h(\Gamma) *'_h$ induces isomorphism between $\mathcal{S}_h(\Gamma)$ and $\mathbf{H}_h(\Gamma)$.
- (b) $\mathbf{H}_h(\Gamma) = \mathcal{H}(\Gamma)$ and $|\det X_J| = 1$ for all $J \in \mathbf{J}(\Gamma)$.
- (c) $\hat{B}_h(\Gamma | -i\underline{\theta}) \neq 0$ for all $\underline{\theta} \in \Theta(\Gamma)$ and $|\det X_J| = 1$ for all $J \in \mathbf{J}(\Gamma)$.
- (d) $h(\underline{\theta}^1 - \underline{\theta}^2) \notin 2\pi i \mathbb{Z}^s \setminus \{0\}$ for all $\underline{\theta}^1, \underline{\theta}^2 \in \Theta(\Gamma)$ and $|\det X_J| = 1$ for all $J \in \mathbf{J}(\Gamma)$.

PROOF. We prove the theorem by showing (d) $\Leftrightarrow$ (c) $\Rightarrow$ (b) $\Rightarrow$ (a) $\Rightarrow$ (c). The equivalence (c) $\Leftrightarrow$ (d) follows from the Lemma 6.1. Corollary 6.1 shows that (c)

implies (b). Assuming (b) we conclude from Theorem 1.1(b), (c) that $\dim \mathcal{S}_h(\Gamma) = \dim \mathcal{H}(\Gamma) = \dim \mathbf{H}_h(\Gamma)$, and (a) follows in this case from Theorem 6.1. Finally assume that (a) holds. Then, $\dim \mathcal{S}_h(\Gamma) = \dim \mathbf{H}_h(\Gamma)$ which in view of Theorem 1.1(b), (c) and Proposition 6.1 implies $\dim \mathcal{S}_h(\Gamma) = \dim \mathcal{H}(\Gamma)$. Referring again to Theorem 1.1(b), (c) we conclude $|\det X_J| = 1$ for all $J \in \mathbf{J}(\Gamma)$.

To show that (c) holds, let $\underline{\theta} \in \Theta(\Gamma)$. By (1.11) and Proposition 2.1, $e_{\underline{\theta}}|_{\mathbf{Z}_h^*} \in \mathcal{S}_h(\Gamma)$. Since we assume (a) $B_h(\Gamma) * u|_{\mathbf{Z}_h^*} \neq 0$, therefore (6.9) gives $\hat{B}_h(\Gamma|-i\underline{\theta}) \neq 0$. Consequently (c) is valid and the proof of the theorem is completed. $\square$

7. Analysis of $\mathbf{H}(\Gamma)$. To simplify the notations we assume throughout this section that $h = 1$ and omit this subscript. Yet, all the results here are valid for arbitrary h with the obvious modifications.

Our main aim is to derive necessary and sufficient conditions for the equality

$$(7.1) \quad \mathbf{H}(\Gamma) = \mathcal{H}(\Gamma).$$

Utilizing these conditions we also obtain a formula for $\dim \mathbf{H}(\Gamma)$.

In order to analyze $\mathbf{H}(\Gamma)$, it suffices, by Theorem 6.1, to examine the range of $\mathcal{S}(\Gamma)$ under $B(\Gamma)*$. To study the action of $B(\Gamma)*$ on $\mathcal{S}(\Gamma)$ we make use of the direct sum decomposition of Theorem 4.3 and investigate the action of $B(\Gamma)*$ on each summand. For this purpose denote for each $\tilde{\underline{\theta}} \in \tilde{\Theta}(\Gamma)$

$$(7.2) \quad \mathcal{S}_{\tilde{\underline{\theta}}} := e_{\tilde{\underline{\theta}}} \mathcal{H}(\tilde{X}_{\tilde{\underline{\theta}}})|_{\mathbf{Z}^*} = \{e_{\tilde{\underline{\theta}}}(\underline{\alpha})p(\underline{\alpha}) \mid p \in \mathcal{H}(\tilde{X}_{\tilde{\underline{\theta}}})\},$$

$$(7.3) \quad \mathbf{H}_{\tilde{\underline{\theta}}} = \{B(\Gamma) * u \mid u \in \mathcal{S}_{\tilde{\underline{\theta}}}\},$$

$$(7.4) \quad [\tilde{\underline{\theta}}] = \{\underline{\theta} \in \Theta(\Gamma) \mid \tilde{\underline{\theta}} - \underline{\theta} \in 2\pi i \mathbb{Z}^s\}.$$

Using these notations, Theorem 4.3 can be written as

$$(7.5) \quad \mathcal{S}(\Gamma) = \bigoplus_{\tilde{\underline{\theta}} \in \tilde{\Theta}(\Gamma)} \mathcal{S}_{\tilde{\underline{\theta}}},$$

while Theorem 6.1 implies

$$(7.6) \quad \mathbf{H}(\Gamma) = \left\langle \bigcup_{\tilde{\underline{\theta}} \in \tilde{\Theta}(\Gamma)} \mathbf{H}_{\tilde{\underline{\theta}}} \right\rangle.$$

Our first result gives significant information on $\mathbf{H}_{\tilde{\underline{\theta}}}$ in terms of $[\tilde{\underline{\theta}}]$.

PROPOSITION 7.1. *For every $\tilde{\underline{\theta}} \in \tilde{\Theta}(\Gamma)$*

$$(7.7) \quad \mathbf{H}_{\tilde{\underline{\theta}}} \subset \bigoplus_{\underline{\theta} \in [\tilde{\underline{\theta}}]} \mathcal{H}(\Gamma_{\underline{\theta}}).$$

PROOF. Let $u(\underline{\alpha}) = e_{\tilde{\underline{\theta}}}(\underline{\alpha})p(\underline{\alpha}) \in \mathcal{S}_{\tilde{\underline{\theta}}}$. By Proposition 6.1 $\mathbf{H}(\Gamma) \subset \mathcal{H}(\Gamma)$ and therefore Theorem 4.2 implies

$$(7.8) \quad B(\Gamma) * u = \sum_{\underline{\theta}' \in \Theta(\Gamma)} e_{\underline{\theta}'}(\underline{x})p_j(\underline{x}), \quad p_j \in \pi.$$

Now, if $\underline{\theta}^{j_0} \notin [\tilde{\theta}]$ then, as in the proof of Lemma 3.1(b), one can find a difference operator Δ (on $\mathbb{Z}^s$) which on the one hand annihilates $u(\cdot)$ and on the other hand is injective on $\mathcal{H}(\Gamma_{\underline{\theta}^{j_0}})$. Application of Δ to both sides of (7.8) gives

$$\sum_{\underline{\theta}^j \in \Theta(\Gamma)} \Delta(e_{\underline{\theta}^j}(\underline{x})p_j(\underline{x})) = B(\Gamma) * \Delta u = 0.$$

Since Δ is injective on $\mathcal{H}(\Gamma_{\underline{\theta}^{j_0}})$ one concludes $p_{j_0} = 0$. $\square$

REMARK 7.1. Note that the only information used in the proof of Proposition 7.1 was that $u(\cdot)$ is an exponential polynomial, while $B(\Gamma) * u$ is a linear combination of exponential polynomials. Thus, replacing $B(\Gamma)$ by an arbitrary compactly supported $\phi \in \mathcal{D}'(\mathbb{R}^s)$ and assuming that $B(\Gamma) * u$ is a linear combination of PE functions one can easily establish a generalization of Proposition 7.1 with respect to ϕ .

An immediate consequence of Proposition 7.1 is the following

COROLLARY 7.1. *Assume $\tilde{\underline{\theta}} \in \tilde{\Theta}(\Gamma)$ and $[\tilde{\theta}] = \emptyset$. Then $\mathbf{H}_{\tilde{\underline{\theta}}} = 0$.*

REMARK 7.2. Corollary 7.1 covers Theorem 4.2 of [DM₃] and Theorem 7.1 of [R₁] as special cases.

REMARK 7.3. In case (6.10) holds we know from (4.13) and Theorem 4.3 that

$$(7.9) \quad \mathcal{S}(\Gamma) = \mathcal{H}(\Gamma) \oplus E(\Gamma),$$

where $E(\Gamma) = \bigoplus_{\tilde{\underline{\theta}} \in \tilde{\Theta}(\Gamma); [\tilde{\theta}] = \emptyset} \mathcal{S}_{\tilde{\underline{\theta}}}$. Thus, by Theorem 6.2 and Corollary 7.1 we see that (7.9) expresses $\mathcal{S}(\Gamma)$ as a direct sum of two subspaces for which $B(\Gamma) *$ is injective on the first and annihilates the other.

Since each $\underline{\theta} \in \Theta(\Gamma)$ lies exactly in a unique equivalence class $[\tilde{\theta}]$, $\tilde{\underline{\theta}} \in \tilde{\Theta}(\Gamma)$, Proposition 7.1 leads to the conclusion

$$\text{COROLLARY 7.2. } \mathbf{H}(\Gamma) = \bigoplus_{\tilde{\underline{\theta}} \in \tilde{\Theta}(\Gamma)} \mathbf{H}_{\tilde{\underline{\theta}}}.$$

The results obtained so far express certain limitations on $\mathbf{H}_{\tilde{\underline{\theta}}}$. The next proposition takes the opposite direction.

PROPOSITION 7.2. *Let $\tilde{\underline{\theta}} \in \tilde{\Theta}(\Gamma)$ and let $\underline{\theta} \in [\tilde{\theta}]$. Then the operator $D^{\tilde{\Gamma}_{\tilde{\underline{\theta}}} \setminus \Gamma_{\underline{\theta}}}$ maps $\mathbf{H}_{\tilde{\underline{\theta}}}$ onto $\mathcal{H}(\Gamma_{\underline{\theta}})$.*

PROOF. In the proof of Proposition 7.2 we make use of the following lemma:

LEMMA 7.1. *For each $K_1 \subset K_2 \subset \Gamma$, the operator $\nabla^{X_{K_1}}$ maps $\mathcal{H}(X_{K_1})$ onto $\mathcal{H}(X_{K_1 \setminus K_2})$.*

The proof of Lemma 7.1 is based on

LEMMA 7.2. *For each $\gamma \in \Gamma$, D^γ maps $\mathcal{H}(\Gamma)$ onto $\mathcal{H}(\Gamma \setminus \gamma)$.*

PROOF OF LEMMA 7.2. It is easy to see that D^γ maps $\mathcal{H}(\Gamma)$ into $\mathcal{H}(\Gamma \setminus \gamma)$. To prove that this map is onto fix $f \in \mathcal{H}(\Gamma \setminus \gamma)$ and let $h > 0$ be such that (6.10) holds. (The existence of such h is obvious; It is also guaranteed by Corollary 4.2.) Let A be any $\Gamma - h$ cell and let $\underline{y} \in A$. By Corollary 6.1 $\mathbf{H}_h(\Gamma \setminus \gamma) = \mathcal{H}(\Gamma \setminus \gamma)$ so there exists $\{c_{\underline{\alpha}}\}_{\underline{\alpha} \in b_h^{\Gamma \setminus \gamma}(\underline{y})}$ such that

$$(7.10) \quad f|_A = \sum_{\underline{\alpha} \in b_h^{\Gamma \setminus \gamma}(\underline{y})} c_{\underline{\alpha}} B_h(\Gamma \setminus \gamma | \cdot - \underline{\alpha})|_A.$$

Making use of (1.15) we see that for every $\underline{\alpha} \in \mathbb{Z}_h^s$

$$\begin{aligned} B_h(\Gamma \backslash \gamma | \cdot - \underline{\alpha}) &= \sum_{j=0}^{\infty} e^{jh\lambda_\gamma} \nabla_h^\gamma B_h(\Gamma \backslash \gamma | \cdot - \underline{\alpha} - jh\underline{x}_\gamma) \\ &= \sum_{j=0}^{\infty} e^{jh\lambda_\gamma} D^\gamma B_h(\Gamma | \cdot - \underline{\alpha} - jh\underline{x}_\gamma). \end{aligned}$$

The last equation when combined with (7.10) shows that

$$f|_A \in \langle \{D^\gamma B_h(\Gamma | \cdot - \underline{\alpha})|_A | \underline{\alpha} \in \mathbb{Z}_h^s\} \rangle,$$

while from [R₁, Corollary 4.3] (or alternately from the proof of Proposition 6.1) we know that $B_h(\Gamma | \cdot - \underline{\alpha})|_A \in \mathcal{H}(\Gamma)$, $\forall \underline{\alpha} \in \mathbb{Z}_h^s$. Hence $f|_A \in \{D^\gamma g | g \in \mathcal{H}(\Gamma)\}$. Since $\mathcal{H}(\Gamma)$ consists of entire functions this last result readily extends to $\mathbb{R}^s$ and Lemma 7.2 is established.

PROOF OF LEMMA 7.1. By Theorem 4.2 $\mathcal{H}(X_{K_2})$ contains only polynomials, whereas Lemma 4.1 and (1.11) show that

$$\nabla^{X_{K_1}} p = 0 \Leftrightarrow D^{X_{K_1}} p = 0, \quad \forall p \in \pi.$$

Therefore $\ker(\nabla^{X_{K_1}}|_{\mathcal{H}(X_{K_2})}) = \ker(D^{X_{K_1}}|_{\mathcal{H}(X_{K_2})})$ and the claim now follows by repeated use of Lemma 7.2.

To prove Proposition 7.2 denote $\Gamma_0 = \tilde{\Gamma}_{\underline{\theta}} \backslash \Gamma_{\underline{\theta}}$ and let $u(\cdot) = e_{\underline{\theta}}(\cdot)p(\cdot) \in \mathcal{S}_{\underline{\theta}}$. By (1.15)

$$(7.11) \quad D^{\Gamma_0}(B(\Gamma) * u) = B(\Gamma \backslash \Gamma_0) * \nabla^{\Gamma_0} u.$$

Since $\Gamma_0 \subset \tilde{\Gamma}_{\underline{\theta}}$ and $\underline{\theta} - \tilde{\underline{\theta}} \in 2\pi i \mathbb{Z}^s$ then

$$(7.12) \quad \nabla^{\Gamma_0} u(\underline{\alpha}) = e_{\underline{\theta}}(\underline{\alpha}) \nabla^{X_{\Gamma_0}} p(\underline{\alpha}) = e_{\underline{\theta}}(\underline{\alpha}) \nabla^{X_{\Gamma_0}} p(\underline{\alpha}).$$

Also, from the definition of Γ_0 we see that $\hat{B}(\Gamma \backslash \Gamma_0 | -i\underline{\theta}) \neq 0$ and therefore we can appeal to Theorem 6.2 to conclude that $\hat{B}(\Gamma \backslash \Gamma_0)^*$ is an automorphism of $\mathcal{H}(\Gamma_{\underline{\theta}})$. Thus (7.11), (7.12) and Lemma 7.1 can be combined to yield

$$\begin{aligned} \text{Range}(D^{\Gamma_0}|_{\mathbf{H}_{\underline{\theta}}}) &= \{D^{\Gamma_0}(B(\Gamma) * u) | u \in \mathcal{S}_{\underline{\theta}}\} \\ &= \{B(\Gamma \backslash \Gamma_0) * \nabla^{\Gamma_0} u | u \in \mathcal{S}_{\underline{\theta}}\} \\ &= \{B(\Gamma \backslash \Gamma_0) * u | u \in e_{\underline{\theta}} \mathcal{H}(X_{\underline{\theta}})\} \\ &= \{B(\Gamma \backslash \Gamma_0)^* f | f \in \mathcal{H}(\Gamma_{\underline{\theta}})\} = \mathcal{H}(\Gamma_{\underline{\theta}}). \end{aligned}$$

This completes the proof of Proposition 7.2. $\square$

Another proof of Lemma 7.1, which does not make use of box splines, is given in [DR].

We now combine together Propositions 7.1 and 7.2 to obtain the following sufficient condition for (7.1).

THEOREM 7.1. Assume that for every distinct $\underline{\theta}^1, \underline{\theta}^2 \in \Theta(\Gamma)$

$$(7.13) \quad \underline{\theta}^1 - \underline{\theta}^2 \notin 2\pi i \mathbb{Z}^s.$$

Then $\mathbf{H}(\Gamma) = \mathcal{H}(\Gamma)$.

PROOF. By Proposition 6.1 $\mathbf{H}(\Gamma) \subset \mathcal{H}(\Gamma)$. To prove the opposite inclusion it is sufficient, in view of Theorem 4.2, to show that for each $\underline{\theta} \in \Theta(\Gamma)$, $\mathcal{H}(\Gamma_{\underline{\theta}}) \subset \mathbf{H}(\Gamma)$. Fix $\underline{\theta} \in \Theta(\Gamma)$ and let $\tilde{\underline{\theta}} \in \tilde{\Theta}(\Gamma)$ be such that $\underline{\theta} \in [\tilde{\underline{\theta}}]$. By (7.13) $\underline{\theta}$ is the *unique* node in $[\tilde{\underline{\theta}}]$ and therefore Proposition 7.1 shows that $\mathbf{H}_{\tilde{\underline{\theta}}} \subset \mathcal{H}(\Gamma_{\underline{\theta}})$. But Proposition 7.2 ensures the existence of a linear map from $\mathbf{H}_{\tilde{\underline{\theta}}}$ onto $\mathcal{H}(\Gamma_{\underline{\theta}})$. Hence $\mathcal{H}(\Gamma_{\underline{\theta}}) = \mathbf{H}_{\tilde{\underline{\theta}}} \subset \mathbf{H}(\Gamma)$ and the desired inclusion now follows. $\square$

PROOF OF THEOREM 1.5. The theorem follows from Lemma 6.1 and Theorem 7.1. $\square$

REMARK 7.4. Note that in view of Lemma 6.1, Theorem 7.1 is stronger than Corollary 6.1.

For later reference we also record the following result which has been obtained in the proof of Theorem 7.1.

COROLLARY 7.3. *Assume $\tilde{\underline{\theta}} \in \tilde{\Theta}(\Gamma)$ and $[\tilde{\underline{\theta}}]$ consists of one element $\underline{\theta} \in \Theta(\Gamma)$. Then $\mathbf{H}_{\tilde{\underline{\theta}}} = \mathcal{H}(\Gamma_{\underline{\theta}})$.*

To advance the analysis of $\mathbf{H}(\Gamma)$ we attach to each $\tilde{\underline{\theta}} \in \tilde{\Theta}(\Gamma)$ the following subspace of $\mathcal{S}_{\tilde{\underline{\theta}}}$

$$(7.14) \quad \mathcal{L}_{\tilde{\underline{\theta}}} = \{u \in \mathcal{S}_{\tilde{\underline{\theta}}} \mid \nabla^{\tilde{\Gamma}_{\tilde{\underline{\theta}}}} u = 0 \ \forall \underline{\theta} \in [\tilde{\underline{\theta}}]\}.$$

By convention if $[\tilde{\underline{\theta}}] = \emptyset$ we set $\mathcal{L}_{\tilde{\underline{\theta}}} = \mathcal{S}_{\tilde{\underline{\theta}}}$.

The importance of $\mathcal{L}_{\tilde{\underline{\theta}}}$ becomes clear as soon as we know

LEMMA 7.3. *For $u \in \mathcal{S}_{\tilde{\underline{\theta}}}$*

$$(7.15) \quad B(\Gamma) * u = 0 \Leftrightarrow u \in \mathcal{L}_{\tilde{\underline{\theta}}}.$$

PROOF. In case $[\tilde{\underline{\theta}}] = \emptyset$, (7.15) follows from Corollary 7.1 and the definition of $\mathcal{L}_{\tilde{\underline{\theta}}}$. Assume $[\tilde{\underline{\theta}}] \neq \emptyset$. From Proposition 7.1 we know that

$$B(\Gamma) * u = \sum_{\underline{\theta}^j \in [\tilde{\underline{\theta}}]} e_{\underline{\theta}^j}(\underline{x}) p_j(\underline{x}).$$

Fix $\underline{\theta}^j \in [\tilde{\underline{\theta}}]$ and denote $\Gamma_j = \tilde{\Gamma}_{\tilde{\underline{\theta}}} \setminus \Gamma_{\underline{\theta}^j}$. Then by (1.15)

$$(7.16) \quad D^{\Gamma_j}(B(\Gamma) * u) = B(\Gamma \setminus \Gamma_j) * \nabla^{\Gamma_j} u.$$

Now, if $u \in \mathcal{L}_{\tilde{\underline{\theta}}}$ then by definition $\nabla^{\Gamma_j} u = 0$. Thus (7.16) implies $D^{\Gamma_j}(B(\Gamma) * u) = 0$. Since $\Gamma_j \cap \Gamma_{\underline{\theta}^j} = \emptyset$ we know (as in the proof of Theorem 4.2) that D^{Γ_j} is injective on $\mathcal{H}(\Gamma_{\underline{\theta}^j})$, and consequently $p_j = 0$. Since j was arbitrary we conclude $B(\Gamma) * u = 0$.

For the converse implication, assume $B(\Gamma) * u = 0$. Then by (7.16) $B(\Gamma \setminus \Gamma_j) * \nabla^{\Gamma_j} u = 0$. But $\nabla^{\Gamma_j} u \in \mathcal{H}(\Gamma_{\underline{\theta}^j})|_{\mathbf{Z}}$, while $\hat{B}(\Gamma \setminus \Gamma_j | -i\underline{\theta}^j) \neq 0$. Hence by appealing to Theorem 6.2 we obtain $\nabla^{\Gamma_j} u = 0$ and since j was arbitrary we conclude $u \in \mathcal{L}_{\tilde{\underline{\theta}}}$. $\square$

Our next result is a necessary and sufficient condition for (7.1).

THEOREM 7.2. $\mathbf{H}(\Gamma)$ is a proper subspace of $\mathcal{H}(\Gamma)$ if and only if there exists $\tilde{\underline{\theta}} \in \tilde{\Theta}(\Gamma)$ which satisfies

$$(7.17) \quad \dim \mathcal{L}_{\tilde{\underline{\theta}}} > |\mathbf{J}(\tilde{\Gamma}_{\tilde{\underline{\theta}}})| - \sum_{\underline{\theta} \in [\tilde{\underline{\theta}}]} |\mathbf{J}(\Gamma_{\underline{\theta}})|.$$

Moreover, for such $\tilde{\underline{\theta}}$ one must have $|\tilde{\underline{\theta}}| > 1$.

PROOF. By Theorem 6.1

$$\mathbf{H}(\Gamma) = \text{Range}(B(\Gamma) * |_{\mathcal{S}(\Gamma)}),$$

while Corollaries 7.1, 7.2 imply that

$$(i) \quad \mathbf{H}(\Gamma) = \bigoplus_{\tilde{\underline{\theta}} \in \tilde{\Theta}(\Gamma)} \mathbf{H}_{\tilde{\underline{\theta}}} = \bigoplus_{\tilde{\underline{\theta}} \in \tilde{\Theta}(\Gamma)} \text{Range}(B(\Gamma) * |_{\mathcal{S}_{\tilde{\underline{\theta}}}}).$$

$$(ii) \quad \mathbf{H}_{\tilde{\underline{\theta}}} \subset \bigoplus_{\underline{\theta} \in [\tilde{\underline{\theta}}]} \mathcal{H}(\Gamma_{\underline{\theta}}).$$

In addition we know that $\mathcal{H}(\Gamma) = \bigoplus_{\underline{\theta} \in \Theta(\Gamma)} \mathcal{H}(\Gamma_{\underline{\theta}})$ while each $\underline{\theta} \in \Theta(\Gamma)$ is contained in a unique equivalence class $[\tilde{\underline{\theta}}]$. Therefore we see that (7.1) fails to hold if and only if for some $\tilde{\underline{\theta}} \in \tilde{\Theta}(\Gamma)$

$$(7.18) \quad \dim \mathbf{H}_{\tilde{\underline{\theta}}} < \dim \left(\bigoplus_{\underline{\theta} \in [\tilde{\underline{\theta}}]} \mathcal{H}(\Gamma_{\underline{\theta}}) \right),$$

and the definition of $\mathbf{H}_{\tilde{\underline{\theta}}}$ gives

$$(7.19) \quad \dim \mathbf{H}_{\tilde{\underline{\theta}}} = \dim \mathcal{S}_{\tilde{\underline{\theta}}} - \dim \ker(B(\Gamma) * |_{\mathcal{S}_{\tilde{\underline{\theta}}}}).$$

Now, by Theorem 1.1(b) $\dim \mathcal{S}_{\tilde{\underline{\theta}}} = \dim \mathcal{H}(\tilde{X}_{\tilde{\underline{\theta}}}) = |\mathbf{J}(\tilde{\Gamma}_{\tilde{\underline{\theta}}})|$ and $\dim \bigoplus_{\underline{\theta} \in [\tilde{\underline{\theta}}]} \mathcal{H}(\Gamma_{\underline{\theta}}) = \sum_{\underline{\theta} \in [\tilde{\underline{\theta}}]} |\mathbf{J}(\Gamma_{\underline{\theta}})|$, and by Lemma 7.3 $\ker(B(\Gamma) * |_{\mathcal{S}_{\tilde{\underline{\theta}}}}) = \mathcal{L}_{\tilde{\underline{\theta}}}$. Substituting all of these into (7.10) we see that (7.18) is equivalent to (7.17).

Finally, when $|\tilde{\underline{\theta}}| = 0, 1$ we know from Corollaries 7.1, 7.3 respectively that $\mathbf{H}_{\tilde{\underline{\theta}}} = \bigoplus_{\underline{\theta} \in [\tilde{\underline{\theta}}]} \mathcal{H}(\Gamma_{\underline{\theta}})$ and therefore such $\tilde{\underline{\theta}}$ cannot satisfy (7.18) or equivalently (7.17). Consequently if $\tilde{\underline{\theta}} \in \tilde{\Theta}(\Gamma)$ satisfies (7.17) we must have $|\tilde{\underline{\theta}}| > 1$. $\square$

The dimension of $\mathbf{H}(\Gamma)$ can be computed with the aid of the following formulae.

THEOREM 7.3.

$$\begin{aligned} \dim \mathbf{H}(\Gamma) &= \dim \mathcal{S}(\Gamma) - \sum_{\tilde{\underline{\theta}} \in \tilde{\Theta}(\Gamma)} \dim \mathcal{L}_{\tilde{\underline{\theta}}} \\ &= |\mathbf{J}(\Gamma)| - \sum_{\substack{\tilde{\underline{\theta}} \in \tilde{\Theta}(\Gamma) \\ |\tilde{\underline{\theta}}| > 1}} \left(\dim \mathcal{L}_{\tilde{\underline{\theta}}} - |\mathbf{J}(\tilde{\Gamma}_{\tilde{\underline{\theta}}})| + \sum_{\underline{\theta} \in [\tilde{\underline{\theta}}]} |\mathbf{J}(\Gamma_{\underline{\theta}})| \right). \end{aligned}$$

PROOF. Since $\mathcal{S}(\Gamma) = \bigoplus_{\tilde{\theta} \in \tilde{\Theta}(\Gamma)} \mathcal{S}_{\tilde{\theta}}$, the first equation is obtained by summing (7.19) over all $\tilde{\theta} \in \tilde{\Theta}(\Gamma)$ and applying Corollary 7.2 and Lemma 7.3. To prove the second equality note that the proof of Theorem 7.2 shows that

$$\begin{aligned} \dim \mathcal{H}(\Gamma) - \dim \mathbf{H}(\Gamma) &= \sum_{\tilde{\theta} \in \tilde{\Theta}(\Gamma)} \left[\left(\sum_{\theta \in [\tilde{\theta}]} \dim \mathcal{H}(\Gamma_{\theta}) \right) - \dim \mathbf{H}_{\tilde{\theta}} \right] \\ &= \sum_{\substack{\tilde{\theta} \in \tilde{\Theta}(\Gamma) \\ |[\tilde{\theta}]| > 1}} \left[\left(\sum_{\theta \in [\tilde{\theta}]} \dim \mathcal{H}(\Gamma_{\theta}) \right) - \dim \mathcal{S}_{\tilde{\theta}} + \dim \mathcal{L}_{\tilde{\theta}} \right]. \end{aligned}$$

Now the desired result follows directly from Theorem 1.1(b).

EXAMPLE 7.1. We use here Theorem 7.2 to show that the sufficient condition for (7.1) established in Theorem 7.1 is not a necessary one.

Let $s = 2$, $|\Gamma| = 3$, $(\mathbf{x}^1, \lambda_1) = ((1, 0), 0)$, $(\mathbf{x}^2, \lambda_2) = ((1, 1), 2\pi i)$, $(\mathbf{x}^3, \lambda_3) = ((1, -1), 0)$. In this case Γ is simple, $\Theta(\Gamma) = \{(0, 0), (0, 2\pi i), (\pi i, \pi i)\}$ and therefore $\mathcal{H}(\Gamma) = \langle 1, e^{2\pi i x_2}, e^{\pi i(x_1+x_2)} \rangle$. On the other hand $\tilde{\Theta}(\Gamma) = \{(0, 0), (\pi i, \pi i)\}$ and some straightforward computation shows that

$$\mathcal{S}_{(0,0)} = \langle 1, x_1, x_2 \rangle, \quad \mathcal{S}_{(\pi i, \pi i)} = \langle e^{\pi i(x_1+x_2)} \rangle.$$

Since $|[(\pi i, \pi i)]| = 1$, then in view of Theorem 7.2, in order to verify (7.1) we only have to check $\mathcal{L}_{(0,0)}$. Since $[(0, 0)] = \{(0, 0), (0, 2\pi i)\}$ then by definition every element of $\mathcal{L}_{(0,0)}$ is a linear polynomial p which satisfies

$$p(\underline{x}) - p(\underline{x} - (1, 1)) = p(\underline{x}) - p(\underline{x} - (1, -1)) = 0.$$

This readily implies that $\mathcal{L}_{(0,0)} = \langle 1 \rangle$ and therefore $\dim \mathcal{L}_{(0,0)} = 1$. Now $\tilde{\Gamma}_{(0,0)} = \Gamma$ and therefore

$$|\mathbf{J}(\tilde{\Gamma}_{(0,0)})| = |\mathbf{J}(\Gamma)| = 3,$$

while

$$\sum_{\theta \in [(0,0)]} |\mathbf{J}(\Gamma_{\theta})| = |\mathbf{J}(\Gamma_{(0,0)})| + |\mathbf{J}(\Gamma_{(0,2\pi i)})| = 1 + 1 = 2.$$

Thus we see that (7.17) is not valid here and Theorem 7.2 implies $\mathbf{H}(\Gamma) = \mathcal{H}(\Gamma)$. Indeed, it can be shown (e.g., by Poisson's summation formula) that

$$\begin{aligned} B(\Gamma) *' (x_1 + x_2) &= 2, & B(\Gamma) *' (x_1 - x_2) &= 2e^{2\pi i x_2}, \\ B(\Gamma) *' e^{\pi i(x_1+x_2)} &= -4e^{\pi i(x_1+x_2)}. \end{aligned}$$

To make use of Theorem 7.2 one needs to compute $\dim \mathcal{L}_{\tilde{\theta}}$ at least for cases when $|[\tilde{\theta}]| > 1$. In the following we show that Theorem 7.2 implies a *necessary* condition for (7.1) based only on the structure of $\Theta(\Gamma)$.

THEOREM 7.4. Assume that there exists $\tilde{\theta} \in \tilde{\Theta}(\Gamma)$ and $K \subset \Gamma$ such that

- (i) $|\Theta(K)| > 1$,
- (ii) $\Theta(K) \subset [\tilde{\theta}]$.

Then $\mathbf{H}(\Gamma)$ is a proper subspace of $\mathcal{H}(\Gamma)$.

PROOF. First assume $\Gamma = K$. Since $\Theta(\Gamma) \subset [\tilde{\theta}]$ it follows that $\tilde{\Gamma}_{\tilde{\theta}} = \Gamma$. But since $|\Theta(\Gamma)| > 1$ then for each $\underline{\theta} \in \Theta(\Gamma)$ $\Gamma \setminus \Gamma_{\underline{\theta}} \neq \emptyset$ and therefore $\nabla^{\tilde{\Gamma}_{\tilde{\theta}} \setminus \Gamma_{\underline{\theta}}} = \nabla^{\Gamma \setminus \Gamma_{\underline{\theta}}}$ annihilates $e_{\tilde{\theta}} \in \mathcal{S}_{\tilde{\theta}}$. This means that $\dim \mathcal{S}_{\tilde{\theta}} \geq 1$. On the other hand

$$|\mathbf{J}(\tilde{\Gamma}_{\tilde{\theta}})| - \sum_{\underline{\theta} \in [\tilde{\theta}]} |\mathbf{J}(\Gamma_{\underline{\theta}})| = |\mathbf{J}(\Gamma)| - \sum_{\underline{\theta} \in \Theta(\Gamma)} |\mathbf{J}(\Gamma_{\underline{\theta}})| = 0.$$

Consequently $\tilde{\theta}$ satisfies (7.17) and the claim follows in this case from Theorem 7.2. If $K \subsetneq \Gamma$ (and $\langle K \rangle = \mathbb{R}^s$) then by the proof above $\mathbf{H}(K) \neq \mathcal{H}(K)$. Choose $f \in \mathcal{H}(K) \setminus \mathbf{H}(K)$. By Lemma 7.2 there exists $g \in \mathcal{H}(\Gamma)$ such that $D^{\Gamma \setminus K} g = f$. We claim that $g \notin \mathbf{H}(\Gamma)$. Indeed, if $g \in \mathbf{H}(\Gamma)$ then $g = B(\Gamma) * u$ where $u(\cdot)$ is some sequence defined on $\mathbb{Z}^s$. In this case (1.15) would imply

$$f = D^{\Gamma \setminus K} g = D^{\Gamma \setminus K} (B(\Gamma) * s) = B(K) * \nabla^{\Gamma \setminus K} u,$$

contradicting the fact that $f \notin \mathbf{H}(K)$. Consequently $g \in \mathcal{H}(\Gamma) \setminus \mathbf{H}(\Gamma)$ and the proof of Theorem 7.4 is completed. $\square$

Finally, we note that in the *univariate* case condition (7.13) is necessary as well:

COROLLARY 7.4. *In case $s = 1$, (7.13) is a necessary and sufficient condition for (7.1).*

PROOF. The sufficiency has been proved in Theorem 7.1. To verify the necessity note that in the present situation each $\gamma \in \Gamma$ is a basis in $\mathbf{J}(\Gamma)$ and therefore determines a unique $\theta_{\gamma} \in \Theta(\Gamma)$. Now, if (7.13) is not valid then there exists $\tilde{\theta} \in \tilde{\Theta}(\Gamma)$ and distinct $\theta^1, \theta^2 \in \Theta(\Gamma)$ such that $\{\theta^1, \theta^2\} \subset [\tilde{\theta}]$. Choose $\{\gamma_1, \gamma_2\} \subset \Gamma$ such that $\theta_{\gamma_j} = \theta^j$, $j = 1, 2$. We conclude that $\Theta(\{\gamma_1, \gamma_2\}) = \{\theta^1, \theta^2\} \subset [\tilde{\theta}]$. Application of Theorem 7.4 with $K = \{\gamma_1, \gamma_2\}$ shows therefore that (7.1) does not hold. $\square$

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